Math 9052B/4152B - Algebraic Topology
Winter 2015
Homology with coefficients

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April 8, 2015

Given a CW-complex X, we know that its cellular chain complex $C_*^{\mathrm{CW}}(X)$ and singular chain complex $C_*(X)$ have isomorphic homology $H_*^{\mathrm{CW}}(X) \cong H_*(X)$. We want to generalize this statement to homology with coefficients. Along the way, we discuss some related material from homological algebra.

1 Direct approach

Proposition 1.1. *Let X be a CW-complex and G an abelian group. Then there is an isomorphism of homology with coefficients $H_*^{\mathrm{CW}}(X; G) \cong H_*(X; G)$. Moreover, this isomorphism is natural with respect to cellular maps $X \to Y$ and with respect to G (and all group homomorphisms).*

Proof. Recall that the isomorphism $H_n^{\mathrm{CW}}(X) \cong H_n(X)$ was obtained by showing that the two surjections illustrated in the diagram

$$H_n^{\mathrm{CW}}(X) \twoheadleftarrow H_n(X_n) \twoheadrightarrow H_n(X)$$

have the same kernel. This was a consequence of the long exact sequences of the pairs (X_k, X_{k-1}), and the fact that the relative homology $H_*(X_k, X_{k-1})$ is concentrated in degree k. Homology with coefficients also has a (natural) long exact sequence associated to any pair, and the relative homology groups

$$\begin{aligned}
H_i(X_k, X_{k-1}; G) &\cong \widetilde{H}_i(X_k/X_{k-1}; G) \\
&\cong \widetilde{H}_i(\bigvee_{k\text{-cells}} S^k; G) \\
&\cong \bigoplus_{k\text{-cells}} \widetilde{H}_i(S^k; G) \\
&\cong \begin{cases} \bigoplus_{k\text{-cells}} G & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}
\end{aligned}$$

are also concentrated in degree k. Therefore, the proof for the case $G = \mathbb{Z}$ works here as well.

The naturality statements follow from naturality of the diagram

$$H_n^{\mathrm{CW}}(X; G) \longleftarrow H_n(X_n; G) \longrightarrow H_n(X; G)$$

with respect to cellular maps $X \to Y$, and with respect to group homomorphisms $G \to G'$. □

2 Approach using chain homotopy

Proposition 2.1. *Let C_* be a (possibly unbounded) chain complex of* free *abelian groups. Then C_* is quasi-isomorphic to its homology, in fact via a quasi-isomorphism $C_* \xrightarrow{\sim} H_*(C_*)$ (as opposed to a zig-zag).*

Proof. Consider[1] the short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{d} B_{n-1} \longrightarrow 0$$

which is split, since B_{n-1} is a free abelian group, being a subgroup of the free abelian group C_{n-1}. Choosing a splitting $C_n \cong Z_n \oplus B_{n-1}$ for each $n \in \mathbb{Z}$, the chain complex C_* is

[1] Credit to Tyler Lawson for this explanation:
http://mathoverflow.net/questions/10974/does-homology-detect-chain-homotopy-equivalence

isomorphic (though not naturally) to the chain complex illustrated here:

$$\begin{array}{cc} \vdots & \text{degree} \\ \downarrow & \\ Z_{n+1} \oplus B_n & n+1 \\ \downarrow & \\ Z_n \oplus B_{n-1} & n \\ \downarrow & \\ Z_{n-1} \oplus B_{n-2} & n-1 \\ \downarrow & \\ \vdots & \end{array}$$

where the differential d_n is given by the inclusion $B_{n-1} \hookrightarrow Z_{n-1}$. Hence, there is an isomorphism of chain complexes $C_* \cong \bigoplus_{n\in\mathbb{Z}} C_*^{(n)}$ where $C_*^{(n)}$ denotes the tiny chain complex

$$\begin{array}{cc} 0 & \\ \downarrow & \\ B_n & n+1 \\ \big\downarrow & \\ Z_n & n \\ \downarrow & \\ 0 & \end{array}$$

concentrated in degrees n and $n+1$. Consider $H_n(C_*)$ as a chain complex concentrated in degree n. The map $\varphi_n\colon C_*^{(n)} \to H_n(C_*)$ given by the quotient map $Z_n \twoheadrightarrow H_n(C_*) = Z_n/B_n$ in degree n is a chain map which is moreover a quasi-isomorphism. These maps assemble into a quasi-isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \varphi_n\colon \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} H_n(C_*) = H_*(C_*).$$

as claimed. □

Recall the following fact from homological algebra.

Theorem 2.2 (Comparison theorem for projective resolutions)**.** *Let $\mathcal{A}$ be an abelian category, and let M be an object of $\mathcal{A}$, viewed as a chain complex concentrated in degree 0. Let P_* be a (non-negatively graded) chain complex of projective objects, with a chain map $f\colon P_* \to M$, and let D_* a (non-negatively graded) chain complex with a quasi-isomorphism $w\colon D_* \xrightarrow{\sim} M$. Then f admits a lift as in the diagram*

$$\begin{array}{ccc} & & D_* \\ & {\scriptstyle \widetilde{f}}\nearrow & \big\downarrow {\scriptstyle w}\ \sim \\ P_* & \xrightarrow[f]{} & M \end{array}$$

which is unique up to chain homotopy.

Proof. [1, Theorem 2.2.6]. □

Example 2.3. In the category $\mathcal{A} = \mathbf{Ab}$ of abelian groups, an object is projective if and only if it is a free abelian group.

Proposition 2.4. *Let C_* and D_* be (possibly unbounded) chain complexes of* free *abelian groups.*

1. *If C_* and D_* have isomorphic homology $H_*(C_*) \cong H_*(D_*)$, then they are chain homotopy equivalent: $C_* \simeq D_*$.*
2. *If $f\colon C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism, then f is a chain homotopy equivalence.*

Proof. 1. Consider decompositions $C_* \cong \bigoplus_{n\in\mathbb{Z}} C_*^{(n)}$ and $D_* \cong \bigoplus_{n\in\mathbb{Z}} D_*^{(n)}$ as in the proof of Proposition 2.1. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes

$$\begin{array}{ccc} & & D_*^{(n)} \\ & {\scriptstyle \widetilde{\varphi_n}}\nearrow & \big\downarrow {\scriptstyle \psi_n}\ \sim \\ C_*^{(n)} & \xrightarrow[\varphi_n]{\sim} & H_n(C_*) \cong H_n(D_*) \end{array}$$

where a lift $\widetilde{\varphi_n}\colon C_*^{(n)} \to D_*^{(n)}$ exists, by Theorem 2.2. Reversing the roles of C_* and D_*, there also exists a lift $\widetilde{\psi_n}\colon D_*^{(n)} \to C_*^{(n)}$. Uniqueness of lifts up to chain homotopy shows that $\widetilde{\psi_n}$ is chain homotopy inverse to $\widetilde{\varphi_n}$. Therefore, the chain map

$$\bigoplus_{n\in\mathbb{Z}} \widetilde{\varphi_n}\colon \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \xrightarrow{\simeq} \bigoplus_{n\in\mathbb{Z}} D_*^{(n)}$$

is a chain homotopy equivalence, with chain homotopy inverse $\bigoplus_{n\in\mathbb{Z}} \widetilde{\psi_n}$.

2. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes

$$\begin{array}{ccc} D_*^{(n)} & \overset{\psi_n}{\dashrightarrow} & C_*^{(n)} \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle\sim} \\ H_n(D_*) & \underset{\cong}{\xrightarrow{H_n(f)^{-1}}} & H_n(C_*) \end{array}$$

where there exists a lift $\psi_n \colon D_*^{(n)} \to C_*^{(n)}$ (unique up to chain homotopy), by Theorem 2.2. These chain maps define a chain map $\psi \colon D_* \to C_*$ via the diagram

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} D_*^{(n)} & \xrightarrow{\bigoplus_{n\in\mathbb{Z}} \psi_n} & \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \\ {\scriptstyle\cong}\big| & & \big|{\scriptstyle\cong} \\ D_* & \xrightarrow{\quad\psi\quad} & C_*. \end{array}$$

One readily checks that the restriction $f|_{C_*^{(n)}} \colon C_*^{(n)} \to D_*$ is chain homotopic to the composite

$$C_*^{(n)} \xrightarrow{f|_{C_*^{(n)}}} D_* \xrightarrow{\text{proj}} D_*^{(n)} \xhookrightarrow{\text{inc}} D_*$$

and that $\psi \colon D_* \to C_*$ is chain homotopy inverse to $f \colon C_* \to D_*$. □

Proposition 2.5. *The relation of chain homotopy is compatible with the tensor product of chain complexes. In other words, if the chain maps* $\varphi, \psi \colon C_* \to D_*$ *are chain homotopic and* $\varphi', \psi' \colon C'_* \to D'_*$ *are chain homotopic, then the chain maps*

$$\varphi \otimes \varphi', \psi \otimes \psi' \colon C_* \otimes C'_* \to D_* \otimes D'_*$$

are chain homotopic.

Proof. Using the factorizations illustrated in the diagram

$$\begin{array}{ccccc} & & D_* \otimes C'_* & & \\ & {}^{\varphi\otimes\mathrm{id}_{C'_*}}\nearrow & & \searrow^{\mathrm{id}_{D_*}\otimes\varphi'} & \\ C_* \otimes C'_* & & \xrightarrow{\qquad\varphi\otimes\varphi'\qquad} & & D_* \otimes D'_* \\ & {}_{\mathrm{id}_{C_*}\otimes\varphi'}\searrow & & \nearrow_{\varphi\otimes\mathrm{id}_{D'_*}} & \\ & & C_* \otimes D'_* & & \end{array}$$

it suffices to show that $\varphi \otimes \mathrm{id}_{C'_*}$ is chain homotopic to $\psi \otimes \mathrm{id}_{C'_*}$. Let $h\colon C_n \to D_{n+1}$ be a chain homotopy from φ to ψ, i.e., such that the equation $\psi - \varphi = dh + hd$ holds.

Let us check that $h \otimes \mathrm{id}_{C'_*}\colon (C_* \otimes C'_*)_n \to (D_* \otimes C'_*)_{n+1}$ is a chain homotopy from $\varphi \otimes \mathrm{id}_{C'_*}$ to $\psi \otimes \mathrm{id}_{C'_*}$. For any $x_i \in C_i$ and $x'_j \in C'_j$, with $i + j = n$, we have

$$\begin{aligned}
&d(h \otimes \mathrm{id}_{C'_*})(x_i \otimes x'_j) + (h \otimes \mathrm{id}_{C'_*})d(x_i \otimes x'_j) \\
&\qquad = d\left(hx_i \otimes x'_j\right) + (h \otimes \mathrm{id}_{C'_*})\left(dx_i \otimes x'_j + (-1)^{|x_i|} x_i \otimes dx'_j\right) \\
&\qquad = dhx_i \otimes x'_j + (-1)^{|hx_i|} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{|x_i|} hx_i \otimes dx'_j \\
&\qquad = dhx_i \otimes x'_j + (-1)^{i+1} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{i} hx_i \otimes dx'_j \\
&\qquad = dhx_i \otimes x'_j + hdx_i \otimes x'_j \\
&\qquad = (dh + hd)x_i \otimes x'_j \\
&\qquad = (\psi - \varphi)x_i \otimes x'_j \\
&\qquad = \psi x_i \otimes x'_j - \varphi x_i \otimes x'_j.
\end{aligned}$$

Therefore the equation

$$d(h \otimes \mathrm{id}_{C'_*}) + (h \otimes \mathrm{id}_{C'_*})d = \psi \otimes \mathrm{id}_{C'_*} - \varphi \otimes \mathrm{id}_{C'_*}$$

holds. □

Corollary 2.6. *If $\varphi\colon C_* \xrightarrow{\simeq} D_*$ and $\varphi'\colon C'_* \xrightarrow{\simeq} D'_*$ are chain homotopy equivalences, then their tensor product*

$$\varphi \otimes \varphi'\colon C_* \otimes C'_* \xrightarrow{\simeq} D_* \otimes D'_*$$

is a chain homotopy equivalence.

Proof. Let $\alpha\colon D_* \to C_*$ and $\alpha'\colon D'_* \to C'_*$ be chain homotopy inverses of φ and φ' respectively. Then

$$\alpha \otimes \alpha'\colon D_* \otimes D'_* \to C_* \otimes C'_*$$

is a chain homotopy inverse of $\varphi \otimes \varphi'$. □

The following proposition says that "any chain complex of free abelian groups will do", as long as it has the correct homology (with coefficients in $\mathbb{Z}$).

Proposition 2.7. *Let X be a space and C_* a chain complex of free abelian groups whose homology is isomorphic to the singular homology of X, i.e., $H_n(C_*) \cong H_n(X)$ holds for all n. Then for any abelian group G and any n, there are isomorphisms $H_n(C_* \otimes G) \cong H_n(X; G)$.*

Proof. The assumption is that the homology C_* is isomorphic to the homology of the singular chain complex $C_*(X)$. By Proposition 2.4, there is a chain homotopy equivalence $\varphi\colon C_* \xrightarrow{\simeq} C_*(X)$. By Corollary 2.6, the chain map

$$\varphi \otimes \mathrm{id}_G\colon C_* \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$$

is a chain homotopy equivalence, in particular a quasi-isomorphism. □

Example 2.8. Let X be a Δ-complex, and $C_*^\Delta(X)$ the associated simplicial chain complex. Then there are isomorphisms $H_n^\Delta(X;G) \cong H_n(X;G)$. Naturality with respect to Δ-maps $X \to Y$ does not follow directly from the first part of Proposition 2.7.

However, recall that the isomorphism $H_n^\Delta(X) \cong H_n(X)$ is induced at the chain level by a quasi-isomorphism $\theta \colon C_*^\Delta(X) \xrightarrow{\sim} C_*(X)$, which is natural with respect to Δ-maps $X \to Y$. By the second part of Proposition 2.4, θ is in fact a chain homotopy equivalence. By Corollary 2.6, the chain map $\theta \otimes \mathrm{id}_G \colon C_*^\Delta(X) \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$ is also a chain homotopy equivalence, and in particular induces isomorphisms $H_n^\Delta(X;G) \cong H_n(X;G)$. These isomorphisms are natural with respect to Δ-maps $X \to Y$, since the chain map θ is.

Example 2.9. Let X be a CW-complex, and $C_*^{\mathrm{CW}}(X)$ the associated cellular chain complex. Then there are isomorphisms $H_n^{\mathrm{CW}}(X;G) \cong H_n(X;G)$. Naturality with respect to cellular maps $X \to Y$ does not follow from Proposition 2.7.

3 Approach using the universal coefficient theorem

Recall the following fact.

Theorem 3.1 (Universal coefficient theorem)**.** *Let C_* be a chain complex of* free *abelian groups, and G an abelian group. Then for each $n \in \mathbb{Z}$, there is a short exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes G \xrightarrow{\times} H_n(C_* \otimes G) \longrightarrow \mathrm{Tor}\,(H_{n-1}(C_*), G) \longrightarrow 0$$

which is natural in C_* and G. Moreover, the sequence is split, though the splitting is not natural.

Here, the map $\times \colon H_n(C_*) \otimes G \to H_n(C_* \otimes G)$ sends $[\alpha] \otimes g$ to the homology class $[\alpha \otimes g]$. The functor Tor denotes $\mathrm{Tor}_1^{\mathbb{Z}}$, just like our tensor product $\otimes$ denotes the tensor product $\otimes_{\mathbb{Z}}$ over the integers.

Corollary 3.2. *If $f \colon C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism between chain complexes of* free *abelian groups, then the map $f \otimes \mathrm{id}_G \colon C_* \otimes G \to D_* \otimes G$ is a quasi-isomorphism.*

Proof. By the universal coefficient theorem, for each $n \in \mathbb{Z}$, f induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(C_*) \otimes G & \xrightarrow{\times} & H_n(C_* \otimes G) & \longrightarrow & \mathrm{Tor}\,(H_{n-1}(C_*), G) & \longrightarrow & 0 \\
 & & {\scriptstyle H_n(f)\otimes G}\downarrow \cong & & {\scriptstyle H_n(f\otimes \mathrm{id}_G)}\downarrow \therefore\cong & & {\scriptstyle \mathrm{Tor}(H_{n-1}(f),\mathrm{id}_G)}\downarrow \cong & & \\
0 & \longrightarrow & H_n(D_*) \otimes G & \xrightarrow{\times} & H_n(D_* \otimes G) & \longrightarrow & \mathrm{Tor}\,(H_{n-1}(D_*), G) & \longrightarrow & 0
\end{array}$$

where the rows are exact. By assumption, $H_n(f)$ and $H_{n-1}(f)$ are isomorphisms, thus so are the downward maps $H_n(f) \otimes G$ and $\operatorname{Tor}(H_{n-1}(f), \mathrm{id}_G)$. By the 5-lemma, the downward map in the middle $H_n(f \otimes \mathrm{id}_G)$ is also an isomorphism. □

Example 3.3. Corollary 3.2 provides an alternate proof that the chain map

$$\theta \otimes \mathrm{id}_G \colon C_*^{\Delta}(X) \otimes G \xrightarrow{\sim} C_*(X) \otimes G$$

is a quasi-isomorphism, as discussed in Example 2.8.

What if an isomorphism in homology does not come from a chain map, as in the cellular homology theorem? Then we can still argue as follows.

Proposition 3.4. *If two (possibly unbounded) chain complexes of* free *abelian groups* C_* *and* D_* *have isomorphic homology* $H_*(C_*) \cong H_*(D_*)$*, then the chain complexes* $C_* \otimes G$ *and* $D_* \otimes G$ *have isomorphic homology.*

Proof. Using the splitting in the universal coefficient theorem, we have (non-natural) isomorphisms:

$$\begin{aligned} H_n(C_* \otimes G) &\cong H_n(C_*) \otimes G \oplus \operatorname{Tor}(H_{n-1}(C_*), G) \\ &\cong H_n(D_*) \otimes G \oplus \operatorname{Tor}(H_{n-1}(D_*), G) \\ &\cong H_n(D_* \otimes G). \end{aligned}$$

□

Alternate proof. By Proposition 2.4, there exists a quasi-isomorphism $\varphi \colon C_* \xrightarrow{\sim} D_*$ (which is in fact a chain homotopy equivalence). By Corollary 3.2, the chain map $\varphi \otimes \mathrm{id}_G \colon C_* \otimes G \xrightarrow{\sim} D_* \otimes G$ is also a quasi-isomorphism. □

Remark 3.5. Section 3 is essentially doing the same thing as Section 2, from a more computational perspective. A key step for proving the universal coefficient theorem is to choose splittings of the short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{d} B_{n-1} \longrightarrow 0$$

like we did in the proof of Proposition 2.1.

References

[1] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)