Math 9052B/4152B - Algebraic Topology
Winter 2015
Homework 5

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Due date: Friday March 27

Problem 1. We know that for each space X, there is an isomorphism $\widetilde{H}_0(X) \oplus \mathbb{Z} \simeq H_0(X)$ whose restriction to the first summand is the inclusion $\widetilde{H}_0(X) \subset H_0(X)$. Show that this *cannot* be made into an isomorphism which is natural in X.

Problem 2. Let (X, x_0) and (Y, y_0) be well-pointed spaces, and consider the inclusion $X \vee Y \hookrightarrow X \times Y$, which embeds $X \vee Y$ as the subspace $X \times \{y_0\} \cup \{x_0\} \times Y \subseteq X \times Y$. Show that for every $n \geq 0$, the map induced on homology

$$H_n(X \vee Y) \to H_n(X \times Y)$$

is injective.

Problem 3. Use the long exact sequence in homology of the pair $(S^1 \times S^1, S^1 \vee S^1)$ to compute the homology $H_*(S^1 \times S^1)$ in every degree.

Problem 4.

(a) Let $i \colon A \to X$ be a cofibration and assume that A is contractible. Show that the quotient map $q \colon X \twoheadrightarrow X/i(A)$ is a homotopy equivalence.

(b) Consider the circle S^1 and any point $x_0 \in S^1$, and consider the subspace

$$A := S^1 \setminus \{x_0\} \subset S^1.$$

Show that the quotient map $q \colon S^1 \twoheadrightarrow S^1/A$ is *not* a homotopy equivalence.

Problem 5. (Variant of Hatcher #2.1.22) Let X be a finite-dimensional CW-complex of dimension d. Prove the following facts about the homology of X.

(a) $H_i(X) = 0$ for $i > d$.

(b) $H_d(X)$ is a free abelian group.

(c) The inclusion of the n-skeleton $X_n \hookrightarrow X$ induces isomorphisms $H_i(X_n) \xrightarrow{\cong} H_i(X)$ for $i < n$ and a surjection $H_n(X_n) \twoheadrightarrow H_n(X)$.

(d) If there are no cells of dimension $n-1$ or $n+1$, then $H_n(X)$ is free with a basis in bijection with the n-cells of X.

(e) If X has a finite number N of n-cells, then $H_n(X)$ is generated by at most N elements.