Math 9052B/4152B - Algebraic Topology
Winter 2015
Homework 3

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Due date: Wednesday February 25

Problem 1. Let X be a CW-complex, with skeleta $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$.

(a) Show that the inclusion $X_0 \to X$ induces a surjection on path components $\pi_0(X_0) \twoheadrightarrow \pi_0(X)$.

(b) Show that for $n \geq 1$, the inclusion $X_n \to X$ induces a bijection on path components $\pi_0(X_n) \xrightarrow{\cong} \pi_0(X)$.

Problem 2. (Real projective space) For each $n \geq 0$, show that the following spaces are homeomorphic.

1. $\mathbb{R}P^n := \mathbb{R}^{n+1} \setminus \{0\}/x \sim \lambda x$ for any non-zero scalar $\lambda \in \mathbb{R}^\times$.
2. $S^n/x \sim -x$, i.e., the sphere modulo the antipodal action.
3. $D^n/x \sim -x$ for every point on the boundary $x \in \partial D^n = S^{n-1}$.
4. The CW-complex with one 0-cell, one 1-cell, ..., one n-cell, where the n-cell is attached via $\varphi \colon S^{n-1} \to \mathbb{R}P^{n-1}$ the quotient map by the antipodal action, as in the description (2). (Note: This description uses the statement inductively.)

Problem 3. (Variant of Hatcher #1.2.9) Let M_g be an orientable surface of genus g, for some $g \geq 0$. Consider $X := M_g \setminus D$, where D is a small open disk about a point. Show that X does not retract onto its boundary circle $\partial X = S^1$.

For the last problem, let us fix some terminology. Given a topological group G, when we say that G acts on a topological space X, by default we mean that G acts *continuously*, i.e., the action map $G \times X \to X$ is continuous.

When G is a discrete group, a G-action on X is continuous if and only if each multiplication $g(-)\colon X \to X$ is continuous (and thus a homeomorphism).

Recall that a group action is called **free** if all the stabilizers are trivial, i.e., for every $x \in X$, the condition $gx = x$ implies $g = 1 \in G$.

Definition. An action of a discrete group G on a space X is called **properly discontinuous** if each $x \in X$ has a neighborhood U such that the set $\{g \in G \mid U \cap gU \neq \emptyset\}$ is finite. (Some authors call such an action *wandering*. The terminology "properly discontinuous" has conflicting uses in the literature.)

Problem 4. (Variant of Hatcher #1.3.23) Let G be a topological group acting on a space X, and let $\pi\colon X \to X/G$ denote the projection, where X/G is endowed with the quotient topology.

(a) Show that π is an open map, i.e, for every open subset $U \subseteq X$, its image $\pi(U)$ is open in X/G.

(b) Assume that G acts freely and properly discontinuously on a Hausdorff space X. Show that the G-action is a "covering space action" in the following sense: each $x \in X$ has a neighborhood U such that the condition $U \cap gU \neq \emptyset$ implies $g = 1$.

(c) If the G-action is a "covering space action", show that the projection $\pi\colon X \to X/G$ is a covering space.