Math 535 - General Topology
Fall 2012
Homework 6, Lecture 10/3

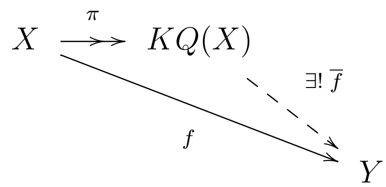
Problem 2. Two points x and y in a topological space X are **topologically distinguishable** if there is an open subset $U \subset X$ that contains one of the points but not the other. A space X is T_0 if any distinct points are topologically distinguishable.

Two points x and y are **topologically indistinguisable** if they are not topologically distinguishable, which amounts to x and y having exactly the same neighborhoods. One readily checks that topological indistinguishability is an equivalence relation on X, which we will denote by $x \sim y$.

The **Kolmogorov quotient** of X is the quotient $KQ(X) := X/\sim$, where topologically indistinguisable points become identified. In particular, X is T_0 if and only if X is homeomorphic to its Kolmogorov quotient.

a. Show that the Kolmogorov quotient satisfies the following universal property.

1. The natural map $\pi\colon X \twoheadrightarrow KQ(X)$ is continuous.
2. $KQ(X)$ is a T_0 space.
3. For any T_0 space Y and continuous map $f\colon X \to Y$, there is a unique continuous map $\overline{f}\colon KQ(X) \to Y$ satisfying $f = \overline{f} \circ \pi$, i.e. making the diagram



commute.

In other words, $KQ(X)$ is the "closest T_0 space which X maps into".

b. Show that X is regular if and only if its Kolmogorov quotient $KQ(X)$ is T_3.

Problem 3. Let (X, d) be a metric space.

a. Let $S \subseteq X$ be a non-empty subset and consider the function $f_S\colon X \to \mathbb{R}$ defined by

$$f_S(x) = d(x, S).$$

Show that f_S is Lipschitz continuous with Lipschitz constant 1, i.e.

$$|f_S(x) - f_S(y)| \leq d(x, y) \text{ for all } x, y \in X.$$

In particular, f_S is continuous.

b. Show that closed subsets of X can be **precisely** separated by functions, i.e. for any $A, B \subset X$ disjoint closed subsets of X, there is a continuous function $f \colon X \to [0,1]$ satisfying

$$\begin{cases} f(a) = 0 \text{ for all } a \in A \\ f(b) = 1 \text{ for all } b \in B \\ f(x) \in (0,1) \text{ for all } x \notin A \cup B. \end{cases}$$

First assume A and B are non-empty. Then treat the case $B = \emptyset$ separately.

Hint: Recall the equivalence $d(x, S) = 0$ if and only if $x \in \overline{S}$.

Remark. A space is called **perfectly normal** if its closed subsets can be precisely separated by functions. A space is called T_6 if it is T_1 and perfectly normal. We have just shown that every metric space is T_6.