Math 535 - General Topology
Fall 2012
Homework 5, Lecture 9/26

Problem 3. Let X be a topological space and consider the set of all continuous functions on X with values in $[0,1]$

$$C := \{f\colon X \to [0,1] \mid f \text{ is continuous}\}.$$

Consider the set $[0,1]^C \cong \prod_{f\in C}[0,1]$ of all functions from C to $[0,1]$, endowed with the product topology. Consider the map

$$\begin{aligned} \varphi\colon X &\to [0,1]^C \\ x &\mapsto (f(x))_{f\in C} \end{aligned}$$

so that $\varphi(x)$ is "evaluation at x".

a. Show that φ is continuous.

b. Show that the closure of the image $\overline{\varphi(X)} \subset [0,1]^C$ is a compact Hausdorff space. (Feel free to assume the axiom of choice!)

c. Show that φ is injective if and only if points of X can be separated by functions, i.e. for any distinct points $x, y \in X$, there is a *continuous* function $f\colon X \to [0,1]$ satisfying $f(x) = 0$ and $f(y) = 1$.

This property is sometimes called **functionally Hausdorff** or **completely Hausdorff**.

d. While we're at it, show that a functionally Hausdorff space is always Hausdorff.

Problem 4. (Willard Exercise 17F.1) A topological space X is **countably compact** if every *countable* open cover of X admits a finite subcover. (In particular, compact always implies countably compact, but not the other way around in general.)

Show that X is countably compact if and only if every sequence in X has a cluster point.

Hint: Recall that compactness can be described in terms of closed sets. Countable compactness has a very similar description in terms of closed sets, which could be useful here.