Math 535 - General Topology
Fall 2012
Homework 5, Lecture 9/24

Problem 1. Let X be a *first-countable* topological space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $y \in X$ a cluster point of this sequence. Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to y.

Remark. This shows that a first-countable compact space is always sequentially compact.

Problem 2. Consider the discrete space with two elements $\{0,1\}$ and consider the space

$$X := \{0,1\}^{\mathcal{P}(\mathbb{N})} \cong \prod_{S\in\mathcal{P}(\mathbb{N})} \{0,1\}$$

with the product topology. Here $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$, i.e. $\mathcal{P}(\mathbb{N}) = \{S \mid S \subseteq \mathbb{N}\}$ is the set of all subsets of $\mathbb{N}$.

One can view X as the set of all functions $f \colon \mathcal{P}(\mathbb{N}) \to \{0,1\}$. In this viewpoint, the canonical projection $p_S \colon X \twoheadrightarrow \{0,1\}$ corresponds to evaluation at S, i.e. sending the function $f \in X$ to $f(S) \in \{0,1\}$.

For each $n \in \mathbb{N}$, consider the element $f_n \in X$ whose components are

$$f_n(S) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S. \end{cases}$$

In other words, f_n is the function "Is n in there?"

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ in X has no convergent subsequence.

Hint: Given any subsequence $(f_{n_k})_{k\in\mathbb{N}}$, show that it does not converge by finding a subset $S \subseteq \mathbb{N}$ such that the sequence $(f_{n_k}(S))_{k\in\mathbb{N}}$ in $\{0,1\}$ does not converge.

Remark. By Tychonoff's theorem, X is compact, so that the sequence $(f_n)_{n\in\mathbb{N}}$ has a cluster point, even though it has no convergent subsequence. Thus X is an example of compact space which is not sequentially compact.