Math 535 - General Topology
Fall 2012
Homework 4, Lecture 9/17

Problem 1. Let $\{\Lambda_\alpha\}_{\alpha \in A}$ be a family of directed set. Show that the product $\prod_{\alpha \in A} \Lambda_\alpha$ becomes a directed set by defining the relation

$$\lambda \leq \lambda' \text{ if } \lambda_\alpha \leq \lambda'_\alpha \text{ in } \Lambda_\alpha \text{ for all } \alpha \in A$$

i.e. the componentwise preorder. (First check that this is indeed a preorder.)

Problem 2. Consider the space $\mathbb{R}^{\mathbb{N}}$ with the *box* topology. Consider the subset

$$Z = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_n > 0 \text{ for all } n \in \mathbb{N}\}$$

and the point $\underline{0} = (0, 0, \ldots)$. We know $\underline{0} \in \overline{Z}$, but now we will find an explicit net in Z that converges to $\underline{0}$.

Consider the directed set $\Lambda := \mathbb{N}^{\mathbb{N}} \cong \prod_{i \in \mathbb{N}} \mathbb{N} = \{(n_1, n_2, \ldots) \mid n_i \in \mathbb{N}\}$ with the componentwise preorder (as in Problem 1).

Consider the net φ in Z indexed by Λ which assigns to the list $\lambda = (n_1, n_2, \ldots)$ the point

$$\varphi(\lambda) = \left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots \right) \in Z.$$

Show that this net φ converges to $\underline{0}$.

Problem 3. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Show that a net $(x_\lambda)_{\lambda \in \Lambda}$ in the product $\prod_{\alpha \in A} X_\alpha$ converges to a point x if and only if for each index $\alpha \in A$, the net $(p_\alpha(x_\lambda))_{\lambda \in \Lambda}$ in X_α converges to $p_\alpha(x)$.

Here $p_\beta \colon \prod_{\alpha \in A} X_\alpha \to X_\beta$ denotes the canonical projection.