Math 535 - General Topology
Fall 2012
Homework 1, Lecture 8/29

Definition. Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is **closed** (with respect to $\mathcal{T}$) if its complement $C^c := X \setminus C$ is open (with respect to $\mathcal{T}$).

Problem 5. Show that the collection of closed subsets of X satisfies the following properties.

1. The empty subset $\emptyset$ and X itself are closed.
2. An arbitrary intersection of closed subsets is closed: C_α closed for all α implies $\bigcap_\alpha C_\alpha$ is closed.
3. A finite union of closed subsets is closed: C, C' closed implies $C \cup C'$ is closed.

Remark. In fact, a collection of subsets satisfies these properties if and **only if** their complements form a topology. Moreover, open subsets and closed subsets determine each other.

Upshot: One might as well define a topology via a collection of "closed subsets" satisfying the three properties above. Their complements then form the topology in question.

Problem 6. Let X be a set. Consider the collection of **cofinite** subsets of X together with the empty subset:

$$\mathcal{T}_{\text{cofin}} := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}.$$

a. Show that $\mathcal{T}_{\text{cofin}}$ is a topology on X, called the cofinite topology.

b. Assuming X is infinite, show that the cofinite topology on X cannot be induced by a metric on X.

Definition. Let X be a set.

- The **discrete** topology on X is the one where all subsets are open:
$$\mathcal{T}_{\text{disc}} = \mathcal{P}(X) = \{U \subseteq X\}.$$
- The **anti-discrete** (or **trivial**) topology on X is the one where only the empty subset and X itself are open:
$$\mathcal{T}_{\text{anti}} = \{\emptyset, X\}.$$

Problem 7. Let D be a discrete topological space and A an anti-discrete topological space.

a. Describe all continuous maps $f \colon D \to X$, where X is an arbitrary topological space.

b. Describe all continuous maps $f \colon X \to A$, where X is an arbitrary topological space.

c. Describe all continuous maps $f \colon A \to X$, where X is a metric space.

Remark. We will come back to the question of mapping *into* a discrete space when discussing the notion of connectedness.