Math 535 - General Topology
Fall 2012
Homework 11, Lecture 11/9

Definition. Let X and Y be normed real vector spaces. A linear map $T\colon X \to Y$ is **bounded** if there exists a constant $C \in \mathbb{R}$ satisfying

$$\|Tx\| \leq C\|x\|$$

for all $x \in X$.

By linearity, this condition is equivalent to the following number being finite:

$$\begin{aligned}\|T\| &:= \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} \\ &= \sup_{\|x\|=1} \|Tx\| \\ &= \sup_{\|x\| \leq 1} \|Tx\|.\end{aligned}$$

The number $\|T\| \in \mathbb{R} \cup \{\infty\}$ is called the **operator norm** of T.

Let

$$\mathcal{L}(X,Y) := \{T\colon X \to Y \mid T \text{ is linear and } \|T\| < \infty\}$$

denote the vector space of bounded linear maps from X to Y. It is a vector space under pointwise addition and scalar multiplication. One readily checks that the assignment $T \mapsto \|T\|$ is indeed a norm on $\mathcal{L}(X,Y)$.

Problem 5. Let $T\colon X \to Y$ be a linear map between normed real vector spaces. Show that the following are equivalent.

1. T is continuous (everywhere).
2. T is continuous at some point $x_0 \in X$.
3. T is continuous at $0 \in X$.
4. T is bounded.

Problem 6. Consider the Banach space

$$l^\infty = \{x \in \mathbb{R}^{\mathbb{N}} \mid \|x\|_\infty < \infty\}$$

with the supremum norm $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$. Consider the linear subspace of lists that are eventually zero:

$$X := \{x \in l^\infty \mid \exists\, N \in \mathbb{N} \text{ such that } x_i = 0 \text{ for all } i > N\} \subset l^\infty.$$

Consider the continuous linear maps $T_n\colon X \to \mathbb{R}$ defined by

$$T_n(x) = nx_n.$$

a. Show that the collection $\{T_n\}_{n \in \mathbb{N}}$ is pointwise bounded but not uniformly bounded.

b. Part (a) implies that X cannot be complete. Show explicitly that X is not complete by exhibiting a Cauchy sequence in X that does not converge in X.