Math 535 - General Topology
Fall 2012
Homework 11, Lecture 11/5

Problem 1. Let X be a topological space.

a. Show that the following properties of a subset $A \subseteq X$ are equivalent.

1. The closure of A in X has empty interior: $\mathrm{int}(\overline{A}) = \emptyset$.
2. For all non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ satisfying $V \cap A = \emptyset$.

A subset $A \subseteq X$ satisfying these equivalent properties is called **nowhere dense** in X.

b. Show that the following properties of the space X are equivalent.

1. Any countable intersection of open dense subsets is dense. In other words, if each $U_n \subseteq X$ is open and dense in X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.
2. Any countable union of closed subsets with empty interior has empty interior. In other words, if each $C_n \subseteq X$ is closed in X and satisfies $\mathrm{int}(C_n) = \emptyset$, then their union satisfies $\mathrm{int}\left(\bigcup_{n=1}^{\infty} C_n\right) = \emptyset$.

A space X satisfying these equivalent properties is called a **Baire space**.

Problem 2. (Willard Exercise 25C) Let X be a Baire space and $f \colon X \to \mathbb{R}$ a continuous real-valued function. Show that for every non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ on which f is bounded.

Edit 11/6: This is a general (and basic) fact about *any* topological space X.