**Math 535 - General Topology**
**Fall 2012**
**Homework 10, Lecture 10/31**

**Problem 3.** Let $X$ be a topological space.

**a.** Let $w, x, y, z \in X$, $\alpha\colon [0,1] \to X$ a path from $w$ to $x$, $\beta\colon [0,1] \to X$ a path from $x$ to $y$, and $\gamma\colon [0,1] \to X$ a path from $y$ to $z$. Show that concatenation of paths is associative up to homotopy, in the following sense:

$$(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma) \text{ rel } \{0,1\}.$$

**b.** Let $\alpha\colon [0,1] \to X$ be a path in $X$ from $x$ to $y$. Denote by $\overline{\alpha}\colon [0,1] \to X$ the **reverse** path of $\alpha$, defined by

$$\overline{\alpha}(s) = \alpha(1-s).$$

Show that $\overline{\alpha}$ is inverse to $\alpha$ up to homotopy, in the following sense:

$$\alpha * \overline{\alpha} \simeq 1_x \text{ rel } \{0,1\}$$

where $1_x\colon [0,1] \to X$ denotes the constant path at $x$.

*Remark.* No need to check the condition $\overline{\alpha} * \alpha \simeq 1_y$ rel $\{0,1\}$, which follows from part (b) applied to the path $\overline{\alpha}$ and observing $\overline{\overline{\alpha}} = \alpha$.

*Remark.* We have earned the right to adopt the notation $\overline{\alpha} = \alpha^{-1}$.

**Definition.** Let $A \subseteq X$ be a subspace of $X$, and denote by $i\colon A \to X$ the inclusion. Then $A$ is called...

- a **retract** of $X$ if there is a continuous map $r\colon X \to A$ satisfying $r \circ i = \mathrm{id}_A$, in other words $r(a) = a$ for all $a \in A$. Such a map $r$ is called a **retraction** from $X$ to $A$.
- a **deformation retract** of $X$ if there is a retraction $r\colon X \to A$ which is moreover a homotopy equivalence, i.e. satisfying $i \circ r \simeq \mathrm{id}_X$.

  Explicitly: There is a homotopy $H\colon X \times [0,1] \to X$ satisfying $H(x,0) = x$ for all $x \in X$, $H(x,1) \in A$ for all $x \in X$, and $H(a,1) = a$ for all $a \in A$.
- a **strong deformation retract** of $X$ if there is a retraction $r\colon X \to A$ which moreover satisfies

  $$i \circ r \simeq \mathrm{id}_X \text{ rel } A.$$

  Explicitly: There is a homotopy $H\colon X \times [0,1] \to X$ satisfying $H(x,0) = x$ for all $x \in X$, $H(x,1) \in A$ for all $x \in X$, and $H(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$.

**Problem 4.** Consider the 2-simplex

$$\Delta^2 := \{(x,y) \in \mathbb{R}^2 \mid x + y \leq 1, x \geq 0, y \geq 0\}$$

and consider the subspace of $\Delta^2$ consisting of points on the coordinate axes

$$A = \{(x,y) \in \Delta^2 \mid x = 0 \text{ or } y = 0\} = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}).$$

Show that $A$ is a strong deformation retract of $\Delta^2$.