Math 535 - General Topology
Additional notes

Martin Frankland

November 28, 2012

1 Evaluation map

Proposition 1.1. *Let X be a locally compact Hausdorff space and Y an arbitrary topological space. Then the evaluation map*

$$e\colon X \times C(X,Y) \to Y$$

given by $e(x,f) = f(x)$ is continuous.

Remark 1.2. The evaluation map is not always continuous.

Exercise 1.3 (Munkres 46.8). Let $\mathcal{T}$ be a topology on the set $C(X,Y)$ making the evaluation map $e\colon X \times C(X,Y) \to Y$ continuous. Then $\mathcal{T}$ contains the compact-open topology.

Now let X, Y, and T be topological spaces, and let $F(X,Y)$ denote the *set of all functions* from X to Y. There is a natural bijection of sets:

$$\varphi\colon F(X \times T, Y) \xrightarrow{\cong} F\left(T, F(X,Y)\right) \tag{1}$$

sending a function $H\colon X \times T \to Y$ to the function $\varphi(H)\colon T \to F(X,Y)$ defined by $\varphi(H)(t) = H(-,t) =: h_t$.

Proposition 1.4. ***(a)*** *If a function $H\colon X \times T \to Y$ is continuous, then the following two conditions hold:*

1. *$h_t\colon X \to Y$ is continuous for all $t \in T$;*
2. *The corresponding function $\varphi(H)\colon T \to C(X,Y)$ is continuous.*

(b) *Assuming X is locally compact Hausdorff, the converse holds as well. In other words, if conditions 1. and 2. hold, then the corresponding function $H\colon X \times T \to Y$ is continuous.*

Proof. **(a)** Homework 13 Problem 2.

(b) Rewriting $H(x,t)$ as

$$\begin{aligned} H(x,t) &= H(-,t)(x) \\ &= \left(\varphi(H)(t)\right)(x) \\ &= e\left(x, \varphi(H)(t)\right) \end{aligned}$$

we see that the function $H\colon X \times T \to Y$ corresponding to $\varphi(H)\colon T \to C(X,Y)$ is the composite

$$\begin{array}{ccc} X \times T & \xrightarrow{\quad H \quad} & Y \\ & {\scriptstyle \mathrm{id}_X \times \varphi(H)} \searrow \quad \nearrow {\scriptstyle e} & \\ & X \times C(X,Y). & \end{array}$$

The map $\varphi(H)\colon T \to C(X,Y)$ is continuous by assumption, and so is $\mathrm{id}_X \times \varphi(H)$. Since X is locally compact Hausdorff, the evaluation map e is continuous (by 1.1), and so is the composite $H = e \circ (\mathrm{id}_X \times \varphi(H))$. □

Interpretation. For any spaces X, Y, T, part (a) ensures that the bijection φ from (1) restricts to a map

$$\varphi\colon C(X \times T, Y) \to C\left(T, C(X,Y)\right) \tag{2}$$

sometimes called the **adjunction map**. This latter φ is always injective, since the original φ was injective.

However, the latter φ is not always surjective. In other words, a family $\{h_t\colon X \to Y\}_{t \in T}$ of continuous maps that vary continuously in the parameter $t \in T$ do not always yield a map $H\colon X \times T \to Y$ which is jointly continuous in both arguments.

Part (b) says that if X is nice enough (e.g. locally compact Hausdorff), then φ is indeed surjective.

Remark 1.5. Proposition 1.4 is useful when trying to show that a map $T \to C(X,Y)$ into a mapping space is continuous. By part (a), it suffices that the corresponding map $X \times T \to Y$ be continuous.