# Math 535 - General Topology
# Additional notes

Martin Frankland

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## 1 Partitions of unity

**Definition 1.1.** Let $X$ be a topological space and $f\colon X \to \mathbb{R}$ a continuous function. The **support** of $f$ is the closed subset

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq X.$$

**Definition 1.2.** A cover $\{U_\alpha\}_{\alpha \in A}$ of a space $X$ is **locally finite** if for all $x \in X$, there is a neighborhood $N_x$ of $x$ that intersects only finitely many of the $U_\alpha$, i.e.

$$N_x \cap U_\alpha \neq \emptyset$$

for finitely many $\alpha \in A$.

Note that the definition applies to any kind of cover, not just open covers.

**Definition 1.3.** A **partition of unity** on a space $X$ is a family of continuous functions $\{\rho_\beta \colon X \to [0,1]\}_{\beta \in B}$ satisfying the following two properties.

1. The family $\{\operatorname{supp} \rho_\beta\}_{\beta \in B}$ is locally finite.
2. $\sum_{\beta \in B} \rho_\beta(x) = 1$ for all $x \in X$.

In particular, the supports $\{\operatorname{supp} \rho_\beta\}_{\beta \in B}$ form a cover of $X$, and for all $x \in X$, $\rho_\beta(x) = 0$ for all except finitely many indices $\beta \in B$.

The partition of unity is **subordinate** to a cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ if for all $\beta \in B$, there is an index $\alpha = \alpha(\beta) \in A$ satisfying

$$\operatorname{supp} \rho_\beta \subseteq U_{\alpha(\beta)}.$$

*Remark* 1.4. One could assume WLOG that the indexing set $B$ is $A$. Indeed, one could add together the functions $\rho_\beta$ supported on the same $U_\alpha$, and consider the constant zero function for each index $\alpha$ that does not appear as $\alpha(\beta)$.

It is useful to have partitions of unity subordinate to any open cover. We will find conditions on a space that make this always possible.

# 2 Paracompactness

**Definition 2.1.** Let $\mathcal{U}$ be a cover of $X$. A **refinement** of $\mathcal{U}$ is a cover $\mathcal{V}$ of $X$ such that every member $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$.

In other words, a cover $\{V_\beta\}_{\beta \in B}$ is a refinement of the cover $\{U_\alpha\}_{\alpha \in A}$ if for all $\beta \in B$, there is an index $\alpha(\beta) \in A$ satisfying

$$V_\beta \subseteq U_{\alpha(\beta)}.$$

*Example* 2.2. $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is a refinement of the cover $\{\mathbb{R}\}$ of $\mathbb{R}$.

*Example* 2.3. $\{(0, \infty)\}$ is a refinement of the cover $\{(n, \infty)\}_{n=0}^{\infty}$ of $(0, \infty)$.

**Definition 2.4.** A topological space $X$ is **paracompact** if every open cover of $X$ admits a locally finite open refinement.

*Remark* 2.5. Some authors (e.g. Bredon, Willard) include the Hausdorff condition in the definition of paracompact. Other authors (e.g. Munkres, Wikipedia) do not assume that the space is Hausdorff. We will *not* assume that the definition of paracompact includes Hausdorff.

*Example* 2.6. Every compact space is paracompact.

*Example* 2.7. $\mathbb{R}^n$ is paracompact. This will follow from 2.13.

**Definition 2.8.** A space is **$\sigma$-compact** if it is a countable union of compact subspaces.

*Example* 2.9. $\mathbb{R}^n$ is $\sigma$-compact, as it is the union of closed balls $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}} \overline{B_k(0)}$.

*Example* 2.10. Any closed subset of $\mathbb{R}^n$ is $\sigma$-compact.

*Example* 2.11. More generally, any closed subset $C \subseteq X$ of a $\sigma$-compact Hausdorff space $X$ is $\sigma$-compact.

*Example* 2.12. Any second-countable manifold is $\sigma$-compact. More generally, any second-countable locally compact space is $\sigma$-compact (c.f. Homework 12 #2).

**Proposition 2.13.** *Every locally compact, $\sigma$-compact, Hausdorff space is paracompact.*

**Proposition 2.14.** *A closed subspace of a paracompact space is paracompact.*

*Proof.* Homework 12 #3. □

**Lemma 2.15.** *If $\{A_i\}_{i \in I}$ is a locally finite collection of subsets $A_i \subseteq X$, then we have*

$$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

**Proposition 2.16.** *Every paracompact Hausdorff space is normal.*

**Proposition 2.17** (Shrinking lemma)**.** *Let $X$ be a paracompact Hausdorff space and $\{U_\alpha\}_{\alpha \in A}$ an open cover. Then there is a locally finite open cover $\{V_\alpha\}_{\alpha \in A}$ satisfying $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in A$.*

*Note that the indexing set $A$ is the same for both open covers. Note also that $V_\alpha$ is allowed to be empty.*

**Theorem 2.18** (Existence of partitions of unity)**.** *Let $X$ be a paracompact Hausdorff space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover. Then there is a partition of unity on $X$ subordinate to $\mathcal{U}$.*

# 3 Applications to manifolds

**Theorem 3.1.** *Any compact Hausdorff manifold can be embedded in $\mathbb{R}^N$ for some $N$.*

*Proof.* Let $M$ be an $m$-dimensional compact Hausdorff manifold. For each $x \in M$, a coordinate chart about $x$ consists of a homeomorphism $\varphi_x \colon U_x \to V_x \subseteq \mathbb{R}^m$ where $U_x$ is an open neighborhood of $x$ and $V_x$ is an open subset of $\mathbb{R}^m$. Since $M$ is compact, the open cover $\{U_x\}_{x\in M}$ admits a finite subcover $\{U_1, \ldots, U_k\}$. Since $M$ is paracompact and Hausdorff, there exists a partition of unity $\{\rho_i\}_{i=1}^k$ with $\operatorname{supp} \rho_i \subseteq U_i$.

For $i = 1, \ldots, k$, define maps $h_i \colon M \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \rho_i(x)\varphi_i(x) & \text{if } x \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

These maps $h_i$ are well defined and continuous. Now define the continuous map

$$\begin{aligned} g \colon M &\to \overbrace{\mathbb{R} \times \ldots \times \mathbb{R}}^{k} \times \overbrace{\mathbb{R}^m \times \ldots \times \mathbb{R}^m}^{k} \cong \mathbb{R}^{k(m+1)} \\ x &\mapsto (\rho_1(x), \ldots, \rho_k(x), h_1(x), \ldots, h_k(x)) \,. \end{aligned}$$

Since $M$ is compact and $\mathbb{R}^N$ is Hausdorff, $g$ is a closed map. To show that $g$ is an embedding, it remains to show that $g$ is injective.

Assume $g(x) = g(y)$ for some $x, y \in M$. Then $\rho_i(x) = \rho_i(y)$ for all $i$. Let $j$ be an index where $\rho_j(x) > 0$ and thus $\rho_j(y) = \rho_j(x) > 0$. We obtain

$$\begin{aligned} & h_j(x) = h_j(y) \\ & \rho_j(x)\varphi_j(x) = \rho_j(y)\varphi_j(y) \\ \Rightarrow\ & \varphi_j(x) = \varphi_j(y) \\ \Rightarrow\ & x = y \end{aligned}$$

since $\varphi_j$ is injective. □

This example illustrates how partitions of unity on a manifold can be useful. Since many interesting manifolds are not compact, it would be useful to know when a (Hausdorff) manifold is paracompact.

**Theorem 3.2.** *Let $M$ be a Hausdorff manifold. Then $M$ is paracompact if and only if each connected component of $M$ is second-countable.*

*Proof.* Let $\{M_i\}_{i\in I}$ be the connected components of $M$. Recall that manifolds are locally path-connected. Therefore $M = \coprod_{i\in I} M_i$ is the coproduct of its connected components (which are the same as its path components).

($\Leftarrow$) Note that any manifold is locally compact. Since $M_i$ is locally compact and second-countable, it is $\sigma$-compact (by 2.12).

Since $M_i$ is locally compact, $\sigma$-compact, and Hausdorff, it is paracompact (by 2.13). Since $M = \coprod_{i\in I} M_i$ is a coproduct of paracompact spaces, it is paracompact (c.f. Homework 12 #4).

($\Rightarrow$) Since $M_i$ is connected, locally compact, paracompact, and Hausdorff, it is $\sigma$-compact (by 3.3).

Since $M_i$ is $\sigma$-compact and every point $x \in M_i$ has a second-countable neighborhood, $M_i$ is second-countable (by 3.4). □

**Proposition 3.3.** *Let $X$ be a connected, locally compact, paracompact, Hausdorff space. Then $X$ is $\sigma$-compact.*

**Proposition 3.4.** *Let $X$ be a $\sigma$-compact space such that every point $x \in X$ has a second-countable neighborhood. Then $X$ is second-countable.*

Here is another application of partitions of unity.

**Proposition 3.5.** *Every paracompact smooth manifold admits a Riemannian metric.*

Yet another application is in defining integration on manifolds. One can define integration within a coordinate chart, and then on the entire manifold using a partition of unity.