Math 535 - General Topology
Additional notes

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1 Introduction to groupoids

Definition 1.1. A **groupoid** is a category in which all morphisms are isomorphisms.

Example 1.2. Let G be a group. Consider the groupoid (also denoted G) with one object $*$ and morphisms

$$\mathrm{Hom}_G(*, *) := G$$

where composition is the multiplication in G.

In fact, a locally small groupoid $\mathcal{G}$ with one object is the same as a group, namely the data of a set $\mathrm{Hom}_{\mathcal{G}}(*, *)$ equipped with a binary operation $\circ$ which is associative and unital and has inverses.

Example 1.3. Let $\{G_s\}_{s \in S}$ be a collection of groups, indexed by some set S. One can form the groupoid $\mathcal{G}$ whose objects are $\mathrm{Ob}(\mathcal{G}) = S$ and morphisms are

$$\mathrm{Hom}_{\mathcal{G}}(s, s') = \begin{cases} \emptyset & \text{if } s \neq s' \\ G_s & \text{if } s = s'. \end{cases}$$

Definition 1.4. Let $\mathcal{G}$ be a locally small groupoid. For any object $X \in \mathrm{Ob}(\mathcal{G})$, the hom-set

$$\mathrm{Hom}_{\mathcal{G}}(X, X)$$

forms a group (under composition), called the **vertex group** of $\mathcal{G}$ at X, or the **automorphism group** of X.

This is an instance of a more general phenomenon.

Definition 1.5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$ an object of $\mathcal{C}$.

An **endomorphism** of X is a morphism $f \colon X \to X$ from X to itself.

An **automorphism** of X is a morphism $f \colon X \to X$ which is also an isomorphism.

Let us denote:

$$\mathrm{Iso}_{\mathcal{C}}(X, Y) := \{f \colon X \to Y \mid f \text{ is an isomorphism}\} \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

$$\mathrm{End}_{\mathcal{C}}(X) := \mathrm{Hom}_{\mathcal{C}}(X, X)$$

$$\mathrm{Aut}_{\mathcal{C}}(X) := \mathrm{Iso}_{\mathcal{C}}(X, X).$$

If $\mathcal{C}$ is locally small, then $\mathrm{End}_{\mathcal{C}}(X)$ is a monoid (under composition), called the **endomorphism monoid** of X, while $\mathrm{Aut}_{\mathcal{C}}(X)$ is a group, called the **automorphism group** of X. Note that

$$\mathrm{Aut}_{\mathcal{C}}(X) = \mathrm{End}_{\mathcal{C}}(X)^{\times}$$

is the group of units of $\mathrm{End}_{\mathcal{C}}(X)$.

Example 1.6. In **Set**, let us denote $\underline{n} = \{1, 2, \ldots, n\}$. Then

$$\mathrm{Aut}_{\mathbf{Set}}(\underline{n}) = \Sigma_n$$

is the symmetric group on n letters, consisting of permutations. The corresponding endomorphism monoid

$$\mathrm{End}_{\mathbf{Set}}(\underline{n}) = \{f \colon \underline{n} \to \underline{n}\}$$

consists of all functions $\underline{n} \to \underline{n}$, most of which are not bijective.

Example 1.7. Let $\mathbb{F}$ be a field. In $\mathbf{Vect}_{\mathbb{F}}$, consider the automorphism group

$$\begin{aligned} \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{F}}}(\mathbb{F}^n) &= GL(n, \mathbb{F}) \\ &= \{\text{invertible } n \times n \text{ matrices with coefficients in } \mathbb{F}\}. \end{aligned}$$

The corresponding endomorphism monoid is

$$\begin{aligned} \mathrm{End}_{\mathbf{Vect}_{\mathbb{F}}}(\mathbb{F}^n) &= M_{n \times n}(\mathbb{F}) \\ &= \{\text{all } n \times n \text{ matrices with coefficients in } \mathbb{F}\}. \end{aligned}$$

2 Subcategories

Definition 2.1. A **subcategory** $\mathcal{D}$ of a category $\mathcal{C}$ consists of a subclass of objects $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ and for all objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, a subclass of morphisms

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

such that morphisms in $\mathcal{D}$ contain identity morphisms id_X for all $X \in \mathrm{Ob}(\mathcal{D})$, and are closed under composition.

In other words, $\mathcal{D}$ is a category in its own right, and the inclusion $\iota \colon \mathcal{D} \to \mathcal{C}$ is a functor.

We write $\mathcal{D} \subseteq \mathcal{C}$ to indicate that $\mathcal{D}$ is a subcategory of $\mathcal{C}$.

Example 2.2. Let **Ab** denote the category of abelian groups and group homomorphisms between them. Then **Ab** is a subcategory of **Gp**, the category of all groups.

Example 2.3. Let **FinSet** denote the category of finite sets and functions between them. Then **FinSet** is a subcategory of **Set**, the category of all sets.

Those examples are misleading, as they are not typical of what a subcategory looks like in general.

Definition 2.4. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is **full** if for all objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphisms in $\mathcal{D}$ satisfy

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

In other words, a full subcategory of $\mathcal{C}$ consists of picking some of the objects, and all morphisms between them.

Example 2.5. The examples $\mathbf{Ab} \subseteq \mathbf{Gp}$ and $\mathbf{FinSet} \subseteq \mathbf{Set}$ are full subcategories.

Example 2.6. Let $\mathbf{Set}_{\mathrm{inj}} \subseteq \mathbf{Set}$ denote the subcategory consisting of all sets and injective functions between them. Then $\mathbf{Set}_{\mathrm{inj}}$ is not a full subcategory of $\mathbf{Set}$, since there exist functions that are not injective.

Example 2.7. Let $\mathbf{CHaus} \subseteq \mathbf{Top}$ be the full subcategory consisting of compact Hausdorff spaces (and all continuous maps between them).

Let $\mathbf{LCH} \subseteq \mathbf{Top}$ denote the subcategory consisting of locally compact Hausdorff spaces and *proper* maps between them. Then $\mathbf{LCH}$ is *not* a full subcategory of $\mathbf{Top}$, since there are continuous maps between locally compact Hausdorff spaces that are not proper. However, considering only proper maps allows us to view the one-point extension as a functor

$$(-)^+ \colon \mathbf{LCH} \to \mathbf{CHaus}.$$

Example 2.8. A map $f \colon (X, d_X) \to (Y, d_Y)$ between metric spaces is **non-expansive** if it satisfies

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$

for all $x, x' \in X$. In other words, f is Lipschitz continuous with Lipschitz constant 1.

Let $\mathbf{Met}$ denote the category of metric spaces and non-expansive maps between them.

Let $\mathbf{Met}_{\mathrm{Lip}}$ denote the category of metric spaces and Lipschitz continuous maps between them. Then $\mathbf{Met}_{\mathrm{Lip}}$ is a subcategory of $\mathbf{Met}$ which is not full, since there are Lipschitz continuous maps whose minimal Lipschitz constant is greater than 1. For example, a dilation $f \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = cx$ for some scalar $c \in \mathbb{R}$ is always Lipschitz continuous with Lipschitz constant $|c|$, but f is non-expansive if and only if the condition $|c| \leq 1$ holds.

Example 2.9. Let $\mathbf{MetTop}$ denote the category of metrizable topological spaces and continuous maps between them. Then $\mathbf{MetTop}$ is a full subcategory of $\mathbf{Top}$.

Note that $\mathbf{Met}$ is *not* a subcategory of $\mathbf{Top}$, because the objects are different. An object of $\mathbf{Met}$ is a metric space (X, d_X), which includes the data of a metric. However, there is a forgetful functor

$$U \colon \mathbf{Met} \to \mathbf{Top}$$

which associates to a metric space (X, d_X) its underlying topological space $(X, \mathcal{T}_{\mathrm{met}})$ endowed with the topology induced by d_X. By definition, this forgetful functor lands in the subcategory of metrizable spaces:

$$\mathbf{Met} \xrightarrow{U} \mathbf{MetTop} \subseteq \mathbf{Top}.$$

Example 2.10. Let $\mathcal{C}$ be a category, and consider the subcategory $\mathcal{C}_{\mathrm{iso}}$ having the same objects as $\mathcal{C}$, but with morphisms

$$\mathrm{Hom}_{\mathcal{C}_{\mathrm{iso}}}(X, Y) = \mathrm{Iso}_{\mathcal{C}}(X, Y).$$

By construction, $\mathcal{C}_{\mathrm{iso}}$ is a groupoid. In fact, it is the maximal subgroupoid of $\mathcal{C}$, in the sense that any subcategory $\mathcal{G} \subseteq \mathcal{C}$ which is a groupoid must be contained in $\mathcal{C}_{\mathrm{iso}}$:

$$\mathcal{G} \subseteq \mathcal{C}_{\mathrm{iso}} \subseteq \mathcal{C}.$$

In fact, that statement can be slightly generalized.

Exercise 2.11. Let $\mathcal{G}$ be a groupoid, $\mathcal{C}$ a category, and $F \colon \mathcal{G} \to \mathcal{C}$ any functor. Show that F factors through the maximal subgroupoid of $\mathcal{C}$:

$$\mathcal{G} \xrightarrow{F} \mathcal{C}_{\mathrm{iso}} \subseteq \mathcal{C}.$$