Math 535 - General Topology
Additional notes

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1 More on categories

1.1 Examples of categories

Example 1.1. The category **Set** of sets and functions between them.

Example 1.2. The category **Gp** of groups and group homomorphisms between them.

Example 1.3. For any field $\mathbb{F}$, the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over $\mathbb{F}$ and $\mathbb{F}$-linear transformations between them.

Example 1.4. The category **Top** of topological spaces and continuous functions between them.

Example 1.5. The homotopy category of spaces $h\mathbf{Top}$ whose objects are topological spaces, and morphisms are homotopy classes of continuous functions:

$$\mathrm{Hom}_{h\mathbf{Top}}(X, Y) := [X, Y] := \mathrm{Hom}_{\mathbf{Top}}(X, Y)/\simeq .$$

Example 1.6. Let G be a group. Consider the category (also denoted G) with one object $*$ and morphisms

$$\mathrm{Hom}_G(*, *) := G$$

where composition is the multiplication in G.

Example 1.7. Let M be a monoid. Consider the category (also denoted M) with one object $*$ and morphisms

$$\mathrm{Hom}_M(*, *) := M$$

where composition is the multiplication in M.

Definition 1.8. A category $\mathcal{C}$ is **locally small** if for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y forms a set, called the **hom-set** from X to Y.

Remark 1.9. A locally small category $\mathcal{C}$ with one object is the same as a monoid, namely the data of a set $\mathrm{Hom}_{\mathcal{C}}(*, *)$ equipped with a binary operation $\circ$ which is associative and unital.

1.2 Isomorphisms

Definition 1.10. A morphism $f\colon X \to Y$ in a category $\mathcal{C}$ is an **isomorphism** if there exists a morphism $g\colon Y \to X$ satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

If such a g exists, then it is unique and called the **inverse** of f, denoted $g = f^{-1}$.

Example 1.11. Isomorphisms in **Set** are invertible functions, which is equivalent to being bijective.

Example 1.12. Isomorphisms in **Gp** are group isomorphisms, which is equivalent to being a bijective group homomorphism.

Example 1.13. Isomorphisms in $\mathbf{Vect}_{\mathbb{F}}$ are linear isomorphisms, which is equivalent to being a bijective linear transformation.

Example 1.14. Isomorphisms in **Top** are homeomorphisms, which is *stronger* than being a bijective continuous map.

Example 1.15. Isomorphisms in h**Top** are homotopy equivalences. This proves in particular that a homotopy inverse (if it exists) is unique up to homotopy.

Example 1.16. Let G be a group. In the one-object category G, every morphism is invertible, with inverse g^{-1} as in the group G.

Example 1.17. Let M be a monoid. In the one-object category M, the isomorphisms are

$$M^{\times} := \{m \in M \mid m \text{ has an inverse in } M\}$$

a.k.a. the **group of units** of the monoid M.

2 Functors

2.1 The basics

Definition 2.1. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment that takes objects of $\mathcal{C}$ to objects of $\mathcal{D}$

$$F\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$

$$F\colon\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}\left(F(X), F(Y)\right)$$

in a way that preserves composition and identities:

$$F(g \circ f) = F(g) \circ F(f)$$
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

Schematically:

$$X \xrightarrow{f} Y \qquad \longmapsto \qquad F(X) \xrightarrow{F(f)} F(Y)$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad (g \circ f : X \to Z) \qquad \longmapsto \qquad F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \quad (F(g \circ f) = F(g) \circ F(f) : F(X) \to F(Z))$$

2.2 Examples of functors

2.2.1 From algebra

Example 2.2. $U\colon \mathbf{Gp} \to \mathbf{Set}$ the underlying set functor, which associates to a group its underlying set.

Remark 2.3. The functor U forgets the group structure of G. For this reason, a functors of that type is often called a *forgetful functor*.

Example 2.4. $F\colon \mathbf{Set} \to \mathbf{Gp}$ the free group functor.

Remark 2.5. A group homomorphism from a free group $F(S) \to H$ is determined by its values on all "letters" $s \in S$, and there is no constraint in the choice of said values. In other words, the data of a group homomorphism $F(S) \to H$ is the same as a function between sets $S \to U(H)$. In fact, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Gp}}(F(S), H) \cong \mathrm{Hom}_{\mathbf{Set}}(S, U(H)).$$

We say that F is **left adjoint** to U, or U is **right adjoint** to F.

2.2.2 From topology

Example 2.6. $U\colon \mathbf{Top} \to \mathbf{Set}$ the underlying set functor, which associates to a topological space $(X, \mathcal{T})$ its underlying set X.

Example 2.7. $\mathrm{Dis}\colon \mathbf{Set} \to \mathbf{Top}$ the discrete space functor, which associates to a set S the topological space $(S, \mathcal{T}_{\mathrm{dis}})$ endowed with the discrete topology.

Remark 2.8. Because every function from a discrete space is continuous, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Top}}\left(\mathrm{Dis}(S), (Y, \mathcal{T})\right) \cong \mathrm{Hom}_{\mathbf{Set}}\left(S, U(Y, \mathcal{T})\right)$$

which exhibits Dis as left adjoint to $U\colon \mathbf{Top} \to \mathbf{Set}$.

Example 2.9. $\mathrm{Anti}\colon \mathbf{Set} \to \mathbf{Top}$ the anti-discrete space functor, which associates to a set S the topological space $(S, \mathcal{T}_{\mathrm{anti}})$ endowed with the anti-discrete topology.

Remark 2.10. Because every function to an anti-discrete space is continuous, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Top}}\left((X,\mathcal{T}),\mathrm{Anti}(S)\right) \cong \mathrm{Hom}_{\mathbf{Set}}\left(U(X,\mathcal{T}),S\right)$$

which exhibits Anti as right adjoint to $U\colon \mathbf{Top} \to \mathbf{Set}$.

Example 2.11. Let **CHaus** denote the category of compact Hausdorff spaces and continuous maps between them. Then the Stone-Čech construction

$$\beta\colon \mathbf{Top} \to \mathbf{CHaus}$$

is a functor (c.f. HW 10 Problem 1). By the universal property of β, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{CHaus}}\left(\beta X, K\right) \cong \mathrm{Hom}_{\mathbf{Top}}\left(X, \iota K\right)$$

which exhibits β as left adjoint to the inclusion functor $\iota\colon \mathbf{CHaus} \to \mathbf{Top}$.

Example 2.12. Let $T_0\mathbf{Top}$ denote the category of T_0 spaces and continuous maps between them. Then the Kolgomorov quotient

$$KQ\colon \mathbf{Top} \to T_0\mathbf{Top}$$

defines a functor (c.f. HW 6 Problem 2). By the universal property of KQ, there is a natural bijection

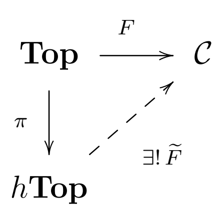
$$\mathrm{Hom}_{T_0\mathbf{Top}}\left(KQ(X), Y\right) \cong \mathrm{Hom}_{\mathbf{Top}}\left(X, \iota Y\right)$$

which exhibits KQ as left adjoint to the inclusion functor $\iota\colon T_0\mathbf{Top} \to \mathbf{Top}$.

2.2.3 From homotopy theory

Example 2.13. Consider the quotient functor $\pi\colon \mathbf{Top} \to h\mathbf{Top}$ which does nothing to objects, and sends a continuous map $f\colon X \to Y$ to its homotopy class $\pi(f) = [f]\colon X \to Y$.

Given any functor $F\colon \mathbf{Top} \to \mathcal{C}$, there is a (unique) factorization $F = \widetilde{F} \circ \pi$ if and only if $F(f)$ depends only on the homotopy class of f, i.e. we have $F(f) = F(f')$ whenever $f \simeq f'$ are homotopic.



Such a functor $F\colon \mathbf{Top} \to \mathcal{C}$ is called a **homotopy functor**.

Example 2.14. The path components functor

$$\pi_0\colon \mathbf{Top} \to \mathbf{Set}$$

is a homotopy functor (c.f. HW 9 Problem 6).

3 Pointed spaces

Definition 3.1. A **pointed space** (or based space) consists of a pair (X, x_0) where X is a topological space and $x_0 \in X$. The point x_0 is called the **basepoint** of X.

Definition 3.2. A **pointed** (or based) map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces is a continuous map that preserves the basepoint i.e. $f(x_0) = y_0$.

Note that a composite $g \circ f$ of pointed maps f and g is pointed, and the identity

$$\mathrm{id}_{(X,x_0)}\colon (X, x_0) \to (X, x_0)$$

is pointed.

Notation 3.3. Let $\mathbf{Top}_*$ denote the category of pointed spaces and pointed maps between them.

Remark 3.4. The disjoint basepoint construction

$$(-)_+\colon \mathbf{Top} \to \mathbf{Top}_*$$

is a functor (c.f. HW 10 Problem 2).

Example 3.5. Assume X is path-connected and $C \subseteq Y$ is the path component of the basepoint $y_0 \in Y$. Then any pointed map

$$f\colon (X, x_0) \to (Y, y_0)$$

must land within C, which implies

$$\mathrm{Hom}_{\mathbf{Top}_*}((X, x_0), (Y, y_0)) \cong \mathrm{Hom}_{\mathbf{Top}_*}((X, x_0), (C, y_0))$$

Definition 3.6. A **pointed** (or based) homotopy between pointed maps $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ is a homotopy

$$F\colon X \times [0, 1] \to Y$$

between f_0 and f_1 such that every intermediate map

$$f_t\colon (X, x_0) \to (Y, y_0)$$

is also pointed, i.e. $F(x_0, t) = y_0$ for all $t \in [0, 1]$.

Example 3.7. A loop in X based at x_0 is a pointed map from the circle

$$\gamma\colon (S^1, *) \to (X, x_0).$$

A pointed homotopy of such loops is a homotopy that remains based at x_0 the entire time.

Exercise 3.8. Show that pointed homotopy $\simeq$ is an equivalence relation on the set of pointed maps between pointed spaces (X, x_0) and (Y, y_0).

Notation 3.9. Denote by

$$[(X, x_0), (Y, y_0)]_* := \mathrm{Hom}_{\mathbf{Top}_*}((X, x_0), (Y, y_0))/{\simeq}$$

the set of pointed homotopy classes of pointed maps from (X, x_0) to (Y, y_0).

Example 3.10. Let $P \simeq *$ be a contractible space. For (unbased) homotopy, we have

$$[P, Y] \cong \pi_0(Y).$$

Assume that P (strongly) deformation retracts onto a point $p_0 \in P$. Then for based homotopy, we have

$$[(P, p_0), (Y, y_0)]_* = \{*\}$$

because every pointed map $(P, p_0) \to (Y, y_0)$ is pointed-homotopic to the constant map at y_0.

Exercise 3.11. Show that pointed homotopy is compatible with composition of pointed maps in the following sense. Given pointed maps

$$f_0, f_1 \colon (X, x_0) \to (Y, y_0)$$
$$g_0, g_1 \colon (Y, y_0) \to (Z, z_0),$$

the conditions $f_0 \simeq f_1$ and $g_0 \simeq g_1$ imply $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Therefore one can compose pointed homotopy classes of pointed maps.

Definition 3.12. The **homotopy category of pointed spaces** $h\mathbf{Top}_*$ has as objects pointed spaces, and morphisms are pointed homotopy classes of pointed maps:

$$\begin{aligned}\mathrm{Hom}_{h\mathbf{Top}_*}((X, x_0), (Y, y_0)) &:= [(X, x_0), (Y, y_0)]_* \\ &= \mathrm{Hom}_{\mathbf{Top}_*}((X, x_0), (Y, y_0))/{\simeq}\,.\end{aligned}$$