Math 535 - General Topology
Additional notes

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October 22, 2012

Proposition 0.1. *Let $A \subseteq X$ be a connected subspace of a topological space X, and $E \subseteq X$ satisfying $A \subseteq E \subseteq \overline{A}$. Then E is connected.*

1 Connected components

Definition 1.1. Consider the relation $\sim$ on X defined by $x \sim y$ if there exists a connected subspace $A \subseteq X$ with $x, y \in A$. Then $\sim$ is an equivalence relation, and the equivalence classes are called the **connected components** of X.

Proposition 1.2.

1. *Let $Z \subseteq X$ be a connected subspace. Then Z lies entirely within one connected component of X.*
2. *Each connected component $C \subseteq X$ is connected.*
3. *Each connected component $C \subseteq X$ is closed in X.*

Remark 1.3. In particular, the connected component C_x of a point $x \in X$ is the largest connected subspace of X that contains x.

Exercise 1.4. A topological space X is **totally disconnected** if its only connected subspaces are singletons $\{x\}$. Show that X is totally disconnected if and only if for all $x \in X$, the connected component C_x of x is the singleton $\{x\}$.

Exercise 1.5. Show that a topological space X is the coproduct of its connected components if and only if the space $X/\sim$ of connected components (with the quotient topology) is discrete.

2 Path-connectedness

Definition 2.1. Let X be a topological space and let $x, y \in X$. A **path** in X from x to y is a continuous map $\gamma\colon [a, b] \to X$ satisfying $\gamma(a) = x$ and $\gamma(b) = y$. Here $a, b \in \mathbb{R}$ satisfy $a < b$.

Definition 2.2. A topological space is **path-connected** is for any $x, y \in X$, there is a path from x to y.

Proposition 2.3. *Let X be a path-connected space. Then X is connected.*

The converse does not hold in general.

Example 2.4 (Topologist's sine curve). The space

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}\} \subset \mathbb{R}^2$$

is path-connected, and therefore connected. By Proposition 0.1, its closure

$$\overline{A} = A \cup (\{0\} \times [-1, 1])$$

is also connected. However, $\overline{A}$ is **not** path-connected.

Proposition 2.5. *Let $f\colon X \to Y$ be a continuous map, where X is path-connected. Then $f(X)$ is path-connected.*

3 Path components

Definition 3.1. Consider the relation $\sim$ on X defined by $x \sim y$ if there exists a path from x to y. Then $\sim$ is an equivalence relation, and the equivalence classes are called the **path components** of X.

Note that there exists a path $\gamma\colon [a, b] \to X$ from x to y if and only if there exists a path $\sigma\colon [0, 1] \to X$ from x to y, taking for example

$$\sigma(t) := \gamma\left(a + t(b - a)\right).$$

We will often assume that the domain of parametrization is $[0, 1]$.

Proof that $\sim$ is an equivalence relation.

1. Reflexivity: The constant path $\gamma\colon [0, 1] \to X$ defined by $\gamma(t) = x$ for all $t \in [0, 1]$ is continuous. This proves $x \sim x$.

2. Symmetry: Assume $x \sim y$, i.e. there is a path $\gamma\colon [0, 1] \to X$ with endpoints $\gamma(0) = x$ and $\gamma(1) = y$. Then $\widetilde{\gamma}\colon [0, 1] \to X$ defined by

$$\widetilde{\gamma}(t) = \gamma(1 - t)$$

   is continuous, since the flip $t \mapsto 1 - t$ is a homeomorphism of $[0, 1]$ onto itself. Moreover $\widetilde{\gamma}$ has endpoints $\widetilde{\gamma}(0) = \gamma(1) = y$ and $\widetilde{\gamma}(1) = \gamma(0) = x$, which proves $y \sim x$.

3. Transitivity: Assume $x \sim y$ and $y \sim z$, i.e. there are paths $\alpha, \beta\colon [0, 1] \to X$ from x to y and from y to z respectively. Define the **concatenation** of the two paths α and β as the path going through α at double speed, followed by β at double speed:

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta\left(2(t - \frac{1}{2})\right) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   This formula is well defined, because for $t = \frac{1}{2}$ we have $\alpha(1) = y = \beta(0)$.

   Moreover, $\alpha * \beta$ is continuous, because its restrictions to the closed subsets $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are continuous, and we have $[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$.

   Finally, $\alpha * \beta$ has endpoints $(\alpha * \beta)(0) = \alpha(0) = x$ and $(\alpha * \beta)(1) = \beta(1) = z$, which proves $x \sim z$. □

Example 3.2. Recall the topologist's sine curve

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}\} \subset \mathbb{R}^2$$

and its closure

$$\overline{A} = A \cup (\{0\} \times [-1, 1])$$

which is connected, and therefore has only one connected component.

However, $\overline{A}$ has exactly two path components: the curve A and the segment $\{0\} \times [-1, 1]$.

Note that A is not closed in $\overline{A}$, so that path components need **NOT** be closed in general, unlike connected components.

Proposition 3.3. *Each path component of X is entirely contained within a connected component of X. In other words, each connected component is a (disjoint) union of path components.*

Proof. If two points x and y are connected by a path $\gamma\colon [a, b] \to X$, then they are both contained in the connected subspace $\gamma([a, b]) \subseteq X$. □

Exercise 3.4. Let $\{A_i\}_{i \in I}$ be a collection of path-connected subspaces of X and $A \subseteq X$ a path-connected subspace satisfying $A \cap A_i \neq \emptyset$ for all $i \in I$. Show that the union $\bigcup_{i \in I} A_i \cup A$ is path-connected.

In particular, if A and B are two path-connected subspaces of X satisfying $A \cap B \neq \emptyset$, then their union $A \cup B$ is path-connected.

Proposition 3.5.

1. *Let $Z \subseteq X$ be a path-connected subspace. Then Z lies entirely within one path component of X.*
2. *Each path component $C \subseteq X$ is path-connected.*

Remark 3.6. In particular, the path component C_x of a point $x \in X$ is the largest path-connected subspace of X that contains x.