Math 535 - General Topology
Additional notes

Martin Frankland

October 17, 2012

1 Proper maps

Definition 1.1. A continuous map $f\colon X \to Y$ between topological spaces is called **proper** if for every compact subspace $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact.

Example 1.2. The projection $p_X\colon X \times Y \to X$ is proper if and only if Y is compact.

Example 1.3. If X is compact and Y is Hausdorff, then every continuous map $f\colon X \to Y$ is both closed and proper.

Definition 1.4. Let X and Y be locally compact Hausdorff spaces, and $f\colon X \to Y$ a continuous map (or any function). The **one-point extension** of f is the function $f^+\colon X^+ \to Y^+$ defined by

$$\begin{cases} f^+(x) = f(x) & \text{if } x \in X \\ f^+(\infty_X) = \infty_Y. \end{cases}$$

Proposition 1.5. *Let $f\colon X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then the one-point extension $f^+\colon X^+ \to Y^+$ is continuous if and only if $f\colon X \to Y$ is proper.*

Corollary 1.6. *Let $f\colon X \to Y$ be a continuous map between locally compact Hausdorff spaces. If f is proper, then f is closed.*

Proposition 1.7 (Sufficient conditions for properness)**.** *Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then f is proper.*

Remark 1.8. The assumption that f is closed cannot be dropped in general. For example, the inclusion $f\colon (0,1] \hookrightarrow [0,1]$ satisfies that $f^{-1}(y)$ is compact for all $y \in [0,1]$, but f is not proper.

2 Localness in the target

Proposition 2.1. *Let $f\colon X \to Y$ be a continuous map between topological spaces. Then f is closed if and only if for all $y \in Y$ and open subset $U \subseteq X$ satisfying $f^{-1}(y) \subseteq U$, there is an open neighborhood V of y satisfying $f^{-1}(V) \subseteq U$.*

Proof. Homework 8 Problem 3. □

Proposition 2.2. *The property of a continuous map $f\colon X \to Y$ being closed is local in the target Y, in the following sense.*

1. **Restriction**: Assume $f\colon X \to Y$ is closed, and let $V \subseteq Y$ be an open subset. Then the restriction
$$f|_{f^{-1}(V)}\colon f^{-1}(V) \to V$$
is closed.

2. **Gluing**: Assume that for all $y \in Y$, there is an open neighborhood V of y such that the restriction
$$f|_{f^{-1}(V)}\colon f^{-1}(V) \to V$$
is closed. Then $f\colon X \to Y$ is closed.

Remark 2.3. Localness in the target can be expressed as follows. The map $f\colon X \to Y$ is closed if and only if for every open cover $\{V_\alpha\}$ of Y, the restrictions

$$f|_{f^{-1}(V_\alpha)}\colon f^{-1}(V_\alpha) \to V_\alpha$$

are closed.

Remark 2.4. The property of $f\colon X \to Y$ being closed is **NOT** local in the domain X. For example, the identity map $\mathrm{id}\colon X \to X$ is always closed, but its restriction $\mathrm{id}|_A\colon A \hookrightarrow X$ to a non-closed subset $A \subset X$ is not closed.

By contrast, the property of continuity is local in the domain X. More precisely, a map $f\colon X \to Y$ is continuous if and only if it is continuous in a neighborhood of every point $x \in X$.

For nice enough spaces, properness is also local in the target.

Proposition 2.5. *The property of a continuous map $f\colon X \to Y$ being proper satisfies the following.*

1. "Restriction" always holds.

2. "Gluing" holds if Y is Hausdorff.

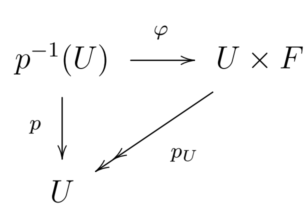
Proof. Homework 8 Problem 4. □

3 Fiber bundles

Definition 3.1. A **fiber bundle** with fiber F consists of a continuous map $p\colon E \to B$ such that for all $b \in B$, there is an open neighborhood $U \subseteq B$ of b and a homeomorphism

$$\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times F$$

which is compatible with the projections, meaning $p_U \circ \varphi = p$, i.e. making the diagram



commute.

In particular, $p\colon E \twoheadrightarrow B$ must be surjective.

The space E is called the **total space** of the bundle, B is called the **base space** of the bundle, and p is called the **projection**.

Example 3.2. Taking $E = B \times F$, the projection map $p_B\colon B \times F \twoheadrightarrow B$ is a fiber bundle with fiber F, called the **trivial bundle**.

By definition, a fiber bundle is "locally trivial". The neighborhood $U \subseteq B$ of b appearing in the definition is called a **trivializing neighborhood** for the bundle.

A homeomorphism $\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times F$ as in the definition is called a **trivialization** of the bundle $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$.

Example 3.3. The Möbius strip M can be viewed as the total space of a (non-trivial) bundle $p\colon M \twoheadrightarrow S^1$ over the circle, with fiber $[-1, 1]$.

See sections 1 - 2.4 of this Wikipedia entry:

`http://en.wikipedia.org/wiki/Fiber_bundle`

for more details.

Proposition 3.4. *Let $p\colon E \twoheadrightarrow B$ be a fiber bundle with fiber F. Then p is proper if and only if the fiber F is compact.*

Proof. ($\Rightarrow$) Pick a point $b \in B$. Since p is proper, the preimage $p^{-1}(b)$ is compact. But in a fiber bundle, we have a homeomorphism $p^{-1}(b) \cong F$, so that F is compact.

($\Leftarrow$) For every $b \in B$, the preimage $p^{-1}(b) \cong F$ is compact by assumption.

For any trivializing neighborhood $U \subseteq B$, the restriction $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a closed map. Indeed, up to homeomorphism, it is the projection $p_U\colon U \times F \twoheadrightarrow U$, which is closed since F is compact. By 2.2, $p\colon E \to B$ is closed.

Since p is closed and the preimage of each point is compact, p is proper (by 1.7). □