Math 535 - General Topology
Additional notes

Martin Frankland

October 10, 2012

1 Locally compact spaces

Definition 1.1. A topological space is **locally compact** if every point $x \in X$ has a compact neighborhood.

Example 1.2. Any compact space is locally compact.

Example 1.3. Any discrete space X is locally compact. Note that X is compact if and only if X is finite.

Example 1.4. Euclidean space $\mathbb{R}^n$ is locally compact.

Example 1.5. Any space locally homeomorphic to $\mathbb{R}^n$ is locally compact. Such a space is called a (topological) **manifold** of dimension n.

Proposition 1.6. *Let X be a locally compact* Hausdorff *topological space. Then for any $x \in X$ and neighborhood V of x, there is a compact neighborhood C of x inside V, i.e. $x \in C \subset V$.*

In other words, compact neighborhoods form a neighborhood basis of every point.

In yet other words, for any open neighborhood U of x, there is an open neighborhood W of x satisfying $x \in W \subseteq \overline{W} \subseteq U$ and $\overline{W}$ is compact.

Remark 1.7. In particular, closed neighborhoods form a neighborhood basis of every point (since compact in Hausdorff is closed). Therefore, a locally compact Hausdorff space is always regular.

In general, a subspace of a locally compact space need not be locally compact.

Example 1.8. Consider $X := \{(0,0)\} \cup \{(x,y) \mid x > 0\} \subset \mathbb{R}^2$. Then X is not locally compact.

Proposition 1.9. *Let X be a locally compact Hausdorff topological space. Then a subspace $A \subseteq X$ is locally compact if and only if it is of the form $A = U \cap F$ for some $U \subseteq X$ open and $F \subseteq X$ closed.*

Proof. ($\Rightarrow$) Consider the equivalent conditions

$$A = U \cap F \text{ for some open } U \subseteq X \text{ and closed } F \subseteq X$$

$$A = U \cap \overline{A} \text{ for some open } U \subseteq X$$

$$A \text{ is open in } \overline{A}.$$

We want to show that if A is locally compact, then A is open in its closure $\overline{A}$.

Let $a \in A$. Since A is locally compact and Hausdorff, there is a subset $O \subset A$ which is open in A, with $a \in O$, and $\overline{O}^A$ (the closure of O in A) is compact. Since O is open in A, there is a subset $V \subseteq X$ open in X satisfying $O = A \cap V$.

We claim $\overline{A} \cap V \subseteq A$. Then note that $\overline{A} \cap V$ is a neighborhood of a in $\overline{A}$ and it lies inside A, which proves that A is open in $\overline{A}$.

The subset $\overline{O}^A = \overline{O} \cap A = \overline{A \cap V} \cap A$ is compact, hence closed in X (since X is Hausdorff). Therefore the inclusion $A \cap V \subseteq \overline{A \cap V} \cap A$ yields

$$\overline{A \cap V} \subseteq \overline{\overline{A \cap V} \cap A} = \overline{A \cap V} \cap A$$

or equivalently $\overline{A \cap V} \subseteq A$.

Moreover, the inclusion $\overline{A} \cap V \subseteq \overline{A \cap V}$ holds, since V is open in X (by 2.1). We conclude

$$\overline{A} \cap V \subseteq \overline{A \cap V} \subseteq A.$$

as claimed. □

Corollary 1.10. *Let X be a locally compact Hausdorff topological space. Then a* dense *subspace $D \subseteq X$ is locally compact if and only if it is open (in X).*

2 Appendix

Proposition 2.1. *Let X be a topological space, $A \subseteq X$ any subset, and $V \subseteq X$ an open subset. Then the inclusion $\overline{A} \cap V \subseteq \overline{A \cap V}$ holds.*

Proof. Let $x \in \overline{A} \cap V$. Since x is in the closure of A, there is a net $(a_\lambda)_{\lambda \in \Lambda}$ in A satisfying $a_\lambda \to x$. Since V is an open neighborhood of x, the net (a_λ) is eventually in V, i.e. there is an index λ_0 guaranteeing $a_\lambda \in V$ whenever $\lambda \geq \lambda_0$. Therefore x is a limit of a net in $A \cap V$, which implies $x \in \overline{A \cap V}$. □

Proof. The inclusion $A \cap V \subseteq A \cap V$ yields

$$\begin{aligned} A \cap V \subseteq A \cap V &\Leftrightarrow A \subseteq V^c \cup (A \cap V) \\ &\Rightarrow \overline{A} \subseteq \overline{V^c \cup (A \cap V)} \\ &\quad = \overline{V^c} \cup \overline{A \cap V} \\ &\quad = V^c \cup \overline{A \cap V} \quad \text{since } V^c \text{ is closed} \\ &\Leftrightarrow \overline{A} \cap V \subseteq \overline{A \cap V}. \end{aligned}$$

□