Math 535 - General Topology
Additional notes

Martin Frankland

October 3, 2012

1 Separation axioms

Definition 1.1. A topological space X is called:

- $\mathbf{T_0}$ or **Kolmogorov** if any distinct points are topologically distinguishable: For $x, y \in X$ with $x \neq y$, there is an open subset $U \subset X$ containing one of the two points but not the other.
- $\mathbf{T_1}$ if any distinct points are separated (i.e. not in the closure of the other): For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$ but $y \notin U_x$, whereas $y \in U_y$ but $x \notin U_y$.
- $\mathbf{T_2}$ or **Hausdorff** if any distinct points can be separated by neighborhoods: For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.
- **regular** if points and closed sets can be separated by neighborhoods: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open subsets $U_x, U_C \subset X$ satisfying $x \in U_x$, $C \subset U_C$, and $U_x \cap U_C = \emptyset$.
- $\mathbf{T_3}$ if it is T_1 and regular.
- **completely regular** if points and closed sets can be separated by functions: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f \colon X \to [0,1]$ satisfying $f(x) = 0$ and $f|_C \equiv 1$.
- $\mathbf{T_{3\frac{1}{2}}}$ or **Tychonoff** if it is T_1 and completely regular.
- **normal** if closed sets can be separated by neighborhoods: For $A, B \subset X$ closed and disjoint, there are open subsets $U, V \subset X$ satisfying $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.
- $\mathbf{T_4}$ if it is T_1 and normal.

There are implications $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ as well as $T_{3\frac{1}{2}} \Rightarrow T_3$. By Urysohn's lemma (see 4.1), the implication $T_4 \Rightarrow T_{3\frac{1}{2}}$ also holds, so that the chain can be written as

$$T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

where each implication is strict (i.e. there are counter-examples to the reverse direction).

2 Equivalent characterizations

Proposition 2.1. *The following are equivalent.*

1. *X is T_1.*

2. *Every singleton $\{x\}$ is closed in X.*

3. *For every $x \in X$, we have*

$$\{x\} = \bigcap_{\substack{\text{all neighborhoods} \\ N \text{ of } x}} N.$$

Proposition 2.2. *The following are equivalent.*

1. *X is T_2.*

2. *The diagonal $\Delta \subseteq X \times X$ is closed in $X \times X$.*

3. *For every $x \in X$, we have*

$$\{x\} = \bigcap_{\substack{\text{closed neighborhoods} \\ C \text{ of } x}} C.$$

Proposition 2.3. *The following are equivalent.*

1. *X is regular.*

2. *For every $x \in X$, any neighborhood of x contains a closed neighborhood of x. In other words, closed neighborhoods form a neighborhood basis of x.*

3. *Given $x \in U$ where U is open, there exists an open $V \subseteq X$ satisfying*

$$x \in V \subseteq \overline{V} \subseteq U.$$

Proposition 2.4. *The following are equivalent.*

1. *X is normal.*

2. *For every $A \subseteq X$ closed, any neighborhood of A contains a closed neighborhood of A.*

3. *Given $A \subseteq U$ where A is closed and U is open, there exists an open $V \subseteq X$ satisfying*

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

3 A few properties

Proposition 3.1. *Behavior of subspaces.*

1. *A subspace of a T_0 space is T_0.*

2. *A subspace of a T_1 space is T_1.*

3. *A subspace of a T_2 space is T_2.*
4. *A subspace of a regular (resp. T_3) space is regular (resp. T_3).*
5. *A subspace of a completely regular (resp. $T_{3\frac{1}{2}}$) space is completely regular (resp. $T_{3\frac{1}{2}}$).*
6. *A* CLOSED *subspace of a normal (resp. T_4) space is normal (resp. T_4).*

Remark 3.2. A subspace of a normal space need *NOT* be normal in general.

Proposition 3.3. *Behavior of (arbitrary) products.*

1. *A product of T_0 spaces is T_0.*
2. *A product of T_1 spaces is T_1.*
3. *A product of T_2 spaces is T_2.*
4. *A product of regular (resp. T_3) spaces is regular (resp. T_3).*
5. *A product of completely regular (resp. $T_{3\frac{1}{2}}$) spaces is completely regular (resp. $T_{3\frac{1}{2}}$).*

Remark 3.4. A product of normal spaces need *NOT* be normal in general, even a finite product.

Proposition 3.5. *Any compact Hausdorff space is T_4. See HW 4 Problem 6.*

Proposition 3.6. *Any metric space is T_4 (in fact T_6). See HW 6 Problem 3.*

4 Urysohn's lemma

Theorem 4.1 (Urysohn's lemma)**.** *Let X be a normal space. Then closed subsets of X can be separated by functions: For $A, B \subseteq X$ closed and disjoint, there is a continuous function $f\colon X \to [0,1]$ satisfying $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

*Such a function is sometimes called an **Urysohn function** for A and B.*

Proof. **Step 1: Construction.**

Since A and B are disjoint, the inclusion $A \subseteq B^c =: U_1$ holds, and note that A is closed and U_1 is open.

Since X is normal, there is an open $U_{\frac{1}{2}}$ satisfying

$$A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_1.$$

Consider the inclusion $A \subseteq U_{\frac{1}{2}}$ where A is closed and $U_{\frac{1}{2}}$ is open. There is an open $U_{\frac{1}{4}}$ satisfying

$$A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}}.$$

Likewise, consider $\overline{U_{\frac{1}{2}}} \subseteq U_1$ where $\overline{U_{\frac{1}{2}}}$ is closed and U_1 is open. There is an open $U_{\frac{3}{4}}$ satisfying

$$\overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U_1.$$

Repeating the process, we obtain for every "dyadic rational" $r = \frac{k}{2^n}$ for some $n \geq 0$ and $0 < k \leq 2^n$ an open subset U_r satisfying

- $A \subseteq U_r$ for all r;
- $\overline{U_r} \subseteq U_s$ whenever $r < s$.

In particular we have $U_r \subseteq U_1 = B^c$ for all r, i.e. every U_r is disjoint from B.

Define the function $f\colon X \to [0,1]$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ belongs to no } U_r \\ \inf\{r \mid x \in U_r\} & \text{otherwise.} \end{cases}$$

Claim: f is an Urysohn function for A and B.

Step 2: Verification.

First, note that the dyadic rationals in $(0,1]$ are dense in $[0,1]$.

The condition $A \subseteq U_r$ for all r implies $f|_A \equiv 0$.

The condition $B \cap U_r = \emptyset$ for all r implies $f|_B \equiv 1$.

It remains to show that f is continuous. This follows from two facts.

Fact A: $x \in \overline{U_r} \Rightarrow f(x) \leq r$. Indeed, the inclusion $\overline{U_r} \subseteq U_s$ holds for all $s > r$, and s can be made arbitrarily close to r.

Fact B: $x \notin U_r \Rightarrow f(x) \geq r$. This is because the set $\{s \mid x \in U_s\}$ is upward closed, and thus cannot contain numbers $q < r$ if r is not in the set. This implies $r \leq \inf\{s \mid x \in U_s\} = f(x)$.

Continuity where $f = 0$.

Assume $f(x) = 0$, and let $\epsilon > 0$. Let r be a dyadic rational in $(0, \epsilon)$. Then we have $x \in U_r$ (by fact B) and $f(y) \leq r < \epsilon$ for all $y \in U_r$ (by fact A). Since U_r is a neighborhood of x, f is continuous at x.

Continuity where $f = 1$.

Assume $f(x) = 1$, and let $\epsilon > 0$. Let r be a dyadic rational in $(1-\epsilon, 1)$. Then we have $x \in \overline{U_r}^c$ (by fact A) and $f(y) \geq r > 1 - \epsilon$ for all $y \in \overline{U_r}^c$ (by fact B). Since $\overline{U_r}^c$ is a neighborhood of x, f is continuous at x.

Continuity where $0 < f < 1$.

Assume $0 < f(x) < 1$, and let $\epsilon > 0$. Take r, s dyadic rationals satisfying

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon.$$

This implies $x \in U_s$ (by fact B) and $x \in \overline{U_r}^c$ (by fact A), in other words $x \in U_s \setminus \overline{U_r}$, which is a neighborhood of x.

Every $y \in U_s$ satisfies $f(y) \leq s$ (by fact A), whereas every $y \in \overline{U_r}^c$ satisfies $f(y) \geq r$ (by fact B), so that the inequality

$$f(x) - \epsilon < r \leq f(y) \leq s < f(x) + \epsilon$$

holds for all $y \in U_s \setminus \overline{U_r}$. This proves continuity of f at x. □

Alternate proof of continuity. Since intervals of the form $[0, \alpha)$ or $(\alpha, 1]$ form a subbasis for the topology of $[0,1]$, it suffices to show that their preimages $f^{-1}[0,\alpha)$ and $f^{-1}(\alpha, 1]$ are open in X.

Consider the equivalent statements:

$$
\begin{aligned}
x \in f^{-1}[0,\alpha) &\Leftrightarrow f(x) < \alpha \\
&\Leftrightarrow \text{ There is a dyadic rational } r < \alpha \text{ satisfying } x \in U_r \\
&\Leftrightarrow x \in \bigcup_{r<\alpha} U_r .
\end{aligned}
$$

This proves the equality

$$f^{-1}[0,\alpha) = \bigcup_{r<\alpha} U_r$$

which is open in X since each U_r is open.

Likewise, consider the equivalent statements:

$$
\begin{aligned}
x \in f^{-1}(\alpha,1] &\Leftrightarrow f(x) > \alpha \\
&\Leftrightarrow \text{ There is a dyadic rational } s > \alpha \text{ satisfying } x \notin U_s \\
&\Leftrightarrow \text{ There is a dyadic rational } r > \alpha \text{ satisfying } x \notin \overline{U_r} \\
&\Leftrightarrow x \in \bigcup_{r>\alpha} \overline{U_r}^c .
\end{aligned}
$$

This proves the equality

$$f^{-1}(\alpha,1] = \bigcup_{r>\alpha} \overline{U_r}^c$$

which is open in X since each $\overline{U_r}^c$ is open. □

Remark 4.2. The result is trivially true if either A or B is empty, but the proof still works!

Remark 4.3. The Urysohn function need not separate A and B *precisely*. In other words, there can be points $x \notin A$ where $f(x) = 0$ and points $y \notin B$ where $f(y) = 1$.