

Math 535 - General Topology

Additional notes

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1 Compactness via nets

Definition 1.1. Let X be a topological space and $(x_\lambda)_{\lambda \in \Lambda}$ a net in X . A point $y \in X$ is a **cluster point** of the net $(x_\lambda)_{\lambda \in \Lambda}$ if for any neighborhood V of y , the net is *often* in V , i.e. for all index $\lambda_0 \in \Lambda$, there is some $\lambda \in \Lambda$ satisfying $\lambda \geq \lambda_0$ and $x_\lambda \in V$.

Proposition 1.2. Let X be a topological space and $(x_\lambda)_{\lambda \in \Lambda}$ a net in X . Then $y \in X$ is a cluster point of $(x_\lambda)_{\lambda \in \Lambda}$ if and only if there is a subnet $(x_{\lambda_\mu})_{\mu \in M}$ converging to y .

Proposition 1.3. A topological space X is compact if and only if every net in X has a cluster point.

2 Zorn's lemma

In order to prove Tychonoff's theorem, we will make use of Zorn's lemma. It is often used to prove the existence of objects with certain maximality properties e.g. maximal interval of existence of solutions to certain differential equations, maximal ideals in a ring, etc.

Definition 2.1. A **partial order** on a set P is a relation \leq which is:

1. reflexive: $x \leq x$ for all $x \in P$;
2. transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$;
3. antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$.

Note that a relation satisfying (1) and (2) is what we previously called a *preorder*.

A **partially ordered set** or **poset** (P, \leq) is a set P equipped with a partial order \leq .

Definition 2.2. A **chain** in a poset P is a subset $C \subseteq P$ which is totally ordered. In other words, any two elements of C are comparable: for all $c, c' \in C$, we have either $c \leq c'$ or $c' \leq c$.

Example 2.3. Let S be a set and consider the poset $\mathcal{P}(S)$ of all subsets of S , ordered by inclusion.

Remark 2.4. Note that reverse inclusion also defines a partial order on $\mathcal{P}(S)$. More generally, given any partial order, its reverse is also a partial order.

Example 2.5. Let $S = \{1, 2, 3, 4, 5\}$ and consider the collection $C = \{\{4\}, \{1, 3, 4\}, \{1, 3, 4, 5\}\} \subset \mathcal{P}(S)$. Then C is a chain in $\mathcal{P}(S)$, i.e. consists of nested subsets of S .

Definition 2.6. An element $m \in P$ in a poset P is **maximal** if no element is greater than m . In other words, the inequality $x \geq m$ implies $x = m$.

Example 2.7. In the poset $\mathcal{P}(S)$, the entire set $S \in \mathcal{P}(S)$ is maximal, and in fact is the only maximal element.

In the poset $\mathcal{P}(S) \setminus \{S\}$, all sets of the form $S \setminus \{s\}$ for some element $s \in S$ are maximal.

In the totally ordered set \mathbb{N} (which is in particular a poset), there is no maximal element.

Definition 2.8. Let $Z \subseteq P$ be a subset of a poset P . An **upper bound** for Z is an element $b \in P$ satisfying $z \leq b$ for all $z \in Z$.

Example 2.9. Consider the poset $\mathcal{P}(S)$ and a collection $\mathcal{C} = \{S_\alpha\}_{\alpha \in A}$ of subsets of S . Then the union $\bigcup_{\alpha \in A} S_\alpha$ is an upper bound for $\mathcal{C} \subseteq \mathcal{P}(S)$, and is in fact the least upper bound for \mathcal{C} .

Theorem 2.10 (Zorn's lemma). *Let P be a non-empty poset such that every chain in P has an upper bound (in P). Then P has a maximal element (i.e. at least one).*

The proof of Zorn's lemma relies on the axiom of choice. In fact, it turns out that Zorn's lemma is equivalent to the axiom of choice.