Math 535 - General Topology
Additional notes

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1 Nets

1.1 Definitions

Definition 1.1. A **preorder** on a set Λ is a relation $\leq$ which is:

1. reflexive: $\lambda \leq \lambda$ for all $\lambda \in \Lambda$;
2. transitive: $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ implies $\lambda_1 \leq \lambda_3$.

Definition 1.2. A **directed set** $(\Lambda, \leq)$ is a set Λ equipped with a preorder $\leq$ such that for every $\lambda_1, \lambda_2 \in \Lambda$, there is some λ_3 satisfying $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

In other words, every finite subset of Λ has an upper bound.

Example 1.3. The natural numbers $\mathbb{N}$ with the usual order $\leq$ form a directed set.

Example 1.4. Let X be a topological space, and $x \in X$. Then the set

$$\Lambda := \mathcal{N}_x = \{N \subseteq X \mid N \text{ is a neighborhood of } x\}$$

ordered by reverse inclusion (i.e. $N_1 \leq N_2$ if $N_2 \subseteq N_1$) is a directed set.

Definition 1.5. Let X be a topological space. A **net** in X is a function $x \colon \Lambda \to X$ from a directed set Λ into X.

We denote values of the net by $x_\lambda := x(\lambda)$ and denote the net by $(x_\lambda)_{\lambda \in \Lambda}$.

Example 1.6. A net in X indexed by $(\mathbb{N}, \leq)$ is a sequence in X.

Definition 1.7. A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space X **converges** to a point $y \in X$ if for all neighborhood N of y, there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in N$ for all $\lambda \geq \lambda_0$.

In words: the net is "eventually" in N.

Convergence will be denoted $x_\lambda \to y$.

1.2 Facts about nets

Proposition 1.8. *Let X be a topological space and $A \subseteq X$ a subset. Then $x \in \overline{A}$ if and only if there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in A which converges to x, i.e. $a_\lambda \to x$.*

In words: the closure of A consists of all limits of nets in A.

Proof. ($\Leftarrow$) Let N be a neighborhood of x. Since (a_λ) converges to x, there is an index $\lambda_0 \in \Lambda$ satisfying $a_\lambda \in N$ for all $\lambda \geq \lambda_0$. In particular, we have $a_{\lambda_0} \in N \cap A \neq \emptyset$. Since N was arbitrary, we conclude $x \in \overline{A}$.

($\Rightarrow$) Let $x \in \overline{A}$. Consider the directed set Λ of all neighborhoods of x, ordered by reverse inclusion. For each $V \in \Lambda$, we have $V \cap A \neq \emptyset$ so we can pick a point $a_V \in V \cap A$. This defines a net $(a_V)_{V\in\Lambda}$ in A. We claim that it converges to x.

Given $W \geq V$, we have $W \subseteq V$ so that $a_W \in W \subseteq V$. In other words, "past the index $V \in \Lambda$, the net is inside the neighborhood $V \subseteq X$", which proves $(a_V)_{V\in\Lambda} \to x$. □

Proposition 1.9. *Let $f\colon X \to Y$ be a map between topological spaces. Then f is continuous at $x \in X$ if and only if for every net $(x_\lambda)_{\lambda\in\Lambda}$ in X with $x_\lambda \to x$, we have $f(x_\lambda) \to f(x)$ in Y.*

Proof. ($\Rightarrow$) Assume $x_\lambda \to x$. We want to show $f(x_\lambda) \to f(x)$.

Let N be a neighborhood of $f(x)$. By continuity of f at x, there is a neighborhood M of x satisfying $f(M) \subseteq N$. By convergence of (x_λ), there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in M$ whenever $\lambda \geq \lambda_0$. Therefore we have $f(x_\lambda) \in f(M) \subseteq N$ whenever $\lambda \geq \lambda_0$, which proves $f(x_\lambda) \to f(x)$.

($\Leftarrow$) Assume f is discontinuous at x, which means there is a neighborhood N of $f(x)$ satisfying $f(M) \nsubseteq N$ for all neighborhoods M of x. For each such neighborhood M, pick a point x_M such that $f(x_M) \notin N$. This defines a net $(x_M)_{M\in\Lambda}$ in X indexed by the directed set Λ of all neighborhoods of x. By construction, the net satisfies $x_M \to x$. However, the net $(f(x_M))_{M\in\Lambda}$ in Y is *never* in N, so in particular $f(x_M) \nrightarrow f(x)$. □

Proposition 1.10 (Uniqueness of limits of nets)**.** *A topological space X is Hausdorff if and only if every net in X has at most one limit. In other words: limits are unique, when they exist.*

Proof. ($\Rightarrow$) Assume X is Hausdorff and $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X with $x_\lambda \to x$ and $x_\lambda \to y$. We want to show $x = y$.

Let U be a neighborhood of x and V a neighborhood of y.

By convergence to x, there is an index $\lambda_1 \in \Lambda$ such that $x_\lambda \in U$ whenever $\lambda \geq \lambda_1$.

By convergence to y, there is an index $\lambda_2 \in \Lambda$ such that $x_\lambda \in V$ whenever $\lambda \geq \lambda_2$.

Let $\lambda_3 \in \Lambda$ be an upper bound for the two indices, i.e. $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. Then we have $x_{\lambda_3} \in U \cap V \neq \emptyset$, so that x and y cannot be separated by neighborhoods. Since X is Hausdorff, this proves $x = y$.

($\Leftarrow$) Assume X is not Hausdorff, which means there exist distinct points $x, y \in X$ which cannot be separated by neighborhoods. In other words, for any neighborhood U of x and neighborhood V of y, we have $U \cap V \neq \emptyset$. Pick a point in the intersection $x_{U,V} \in U \cap V$. This defines a net $(x_{U,V})_{(U,V)\in\Lambda}$ in X indexed by the directed set $\Lambda = \mathcal{N}_x \times \mathcal{N}_y$ of pairs of neighborhoods of x and y respectively.

We show that this net converges to both x and y. Let N be a neighborhood of x. For every index $(U, V) \geq (N, X)$, we have

$$x_{U,V} \in U \cap V \subseteq U \subseteq N$$

which proves $x_{U,V} \to x$. Likewise, we have $x_{U,V} \to y$. □

1.3 Subnets

If nets are meant to generalize sequences, what would be the generalization of subsequences to nets?

Definition 1.11. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. A **subnet** of $(x_\lambda)_{\lambda\in\Lambda}$ is a net $(x_{\lambda_\mu})_{\mu\in M}$ for some directed set M, i.e. the composite

$$M \xrightarrow{\varphi} \Lambda \xrightarrow{x} X$$

where we write $\lambda_\mu := \varphi(\mu)$, and the function $\varphi\colon M \to \Lambda$ is *non-decreasing* and *cofinal*.

Non-decreasing means: $\mu_1 \leq \mu_2 \Rightarrow \varphi(\mu_1) \leq \varphi(\mu_2)$.

Cofinal means that the function will eventually "pass" any index, i.e. for all $\lambda \in \Lambda$, there is some $\mu \in M$ such that $\varphi(\mu) \geq \lambda$.

Example 1.12. The function $\varphi\colon \mathbb{N} \to \mathbb{N}$ defined by $\varphi(k) = 5k$ is non-decreasing and cofinal. Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet

$$(x_{n_k})_{k\in\mathbb{N}} = (x_5, x_{10}, x_{15}, \ldots)$$

where we write $n_k := \varphi(k)$. Note that this is a subsequence.

Example 1.13. The function $\varphi\colon \mathbb{N} \to \mathbb{N}$ defined by $\varphi(k) = \lceil \frac{k}{2} \rceil$ is non-decreasing and cofinal. (Here the brackets denote the ceiling function, which rounds up to the nearest integer.) Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet

$$(x_{n_k})_{k\in\mathbb{N}} = (x_1, x_1, x_2, x_2, x_3, x_3, \ldots).$$

Note that this is *not* a subsequence.

Example 1.14. A function $\varphi\colon \mathbb{N} \to \mathbb{N}$ is cofinal if and only if it is unbounded. Thus a subnet $(x_{n_k})_{k\in\mathbb{N}}$ of a sequence which is still indexed by $\mathbb{N}$ is almost a subsequence, except that indices n_k are allowed to be repeated finitely many times, as in example 1.13.

In contrast, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is defined as having strictly increasing indices: $k_1 < k_2$ implies $n_{k_1} < n_{k_2}$.

Note that a subnet of a sequence can also be indexed by any directed set, not just $\mathbb{N}$.

Example 1.15. The function $\varphi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $\varphi(k, l) = 2k + 4l$ is non-decreasing and cofinal. Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet $(x_{k,l})_{(k,l)\in\mathbb{N}\times\mathbb{N}}$ with values $x_{k,l} := x_{\varphi(k,l)} = x_{2k+4l}$.