Math 535 - General Topology
Additional notes

Martin Frankland

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1 Homeomorphisms

Definition 1.1. A map $f\colon X \to Y$ between topological spaces is a **homeomorphism** if it is continuous, invertible (i.e. bijective), and its inverse $f^{-1}\colon Y \to X$ is also continuous.

2 Neighborhoods

Definition 2.1. Let X be a topological space. A **neighborhood** of a point $x \in X$ is a subset $N \subseteq X$ such that there is an open U satisfying $x \in U \subseteq N$.

3 Bases and subbases

Definition 3.1. Let $(X, \mathcal{T})$ be a topological space. A **basis** for the topology $\mathcal{T}$ of X is a collection $\mathcal{B}$ of subsets of X satisfying

$$\mathcal{T} = \left\{ \bigcup_{\alpha} B_\alpha \mid B_\alpha \in \mathcal{B} \right\}$$

i.e. open sets are precisely unions of members of $\mathcal{B}$.

Exercise 3.2. Let X be a set. Show that a collection $\mathcal{B}$ of subsets of X is a basis for some topology on X if and only if $\mathcal{B}$ satisfies the following conditions:

1. $\mathcal{B}$ covers X, i.e. $\bigcup_{B \in \mathcal{B}} B = X$.
2. Finite intersections are unions: For any $B, B' \in \mathcal{B}$, we have $B \cap B' = \bigcup_\alpha B_\alpha$ for some family $\{B_\alpha\}$ of members of $\mathcal{B}$.

Definition 3.3. Let $(X, \mathcal{T})$ be a topological space. A **subbasis** for the topology $\mathcal{T}$ of X is a collection $\mathcal{S}$ of subsets of X satisfying

$$\mathcal{T} := \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}$$

i.e. finite intersections of members of $\mathcal{S}$ form a basis for the topology.

4 Comparing topologies

For a given set X, topologies on X can be partially ordered by inclusion.

Definition 4.1. Let X be a set, and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X. We say $\mathcal{T}_1$ is **smaller** than $\mathcal{T}_2$, denoted $\mathcal{T}_1 \leq \mathcal{T}_2$, if the inclusion $\mathcal{T}_1 \subseteq \mathcal{T}_2$ holds, viewed as subsets of the power set $\mathcal{P}(X)$. In other words, every $\mathcal{T}_1$-open is also $\mathcal{T}_2$-open.

One can also say that $\mathcal{T}_2$ is **larger** than $\mathcal{T}_1$.

Some references say that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, while $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$.

Remark 4.2. The anti-discrete topology $\mathcal{T}_{\text{anti}} = \{\emptyset, X\}$ is the least element in that partial order, whereas the discrete topology $\mathcal{T}_{\text{dis}} = \mathcal{P}(X)$ is the greatest element. In other words, the inequalities

$$\mathcal{T}_{\text{anti}} \leq \mathcal{T} \leq \mathcal{T}_{\text{dis}}$$

hold for any topology $\mathcal{T}$ on X.

Remark 4.3. By definition, the inequality $\mathcal{T}_1 \leq \mathcal{T}_2$ holds if and only if the identity function

$$\text{id}\colon (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$$

is continuous. Note the reversal, mapping "from fine to coarse".

The poset of topologies on X has arbitrary meets (infima), described explicitly in the following proposition.

Proposition 4.4. *Let $\{\mathcal{T}_\beta\}$ be a family of topologies on X. Then the intersection $\bigcap_\beta \mathcal{T}_\beta$ is a topology on X, and therefore the infimum of the family $\{\mathcal{T}_\beta\}$.*

Proof. Exercise. □

Remark 4.5. If we consider an empty family of topologies, then their intersection is

$$\bigcap_\beta \mathcal{T}_\beta = \mathcal{P}(X) = \mathcal{T}_{\text{dis}}$$

which is a topology on X. Thus the proposition also holds in that case.

Definition 4.6. Let X be a set and $\mathcal{S}$ be a collection of subsets of X. The **topology generated by** $\mathcal{S}$ (if it exists) is the smallest topology $\mathcal{T}_\mathcal{S}$ containing $\mathcal{S}$. In other words, it satisfies $\mathcal{S} \subseteq \mathcal{T}_\mathcal{S}$ and for any other topology $\mathcal{T}'$ containing $\mathcal{S}$, we have $\mathcal{T}_\mathcal{S} \leq \mathcal{T}'$.

Note that this universal property makes $\mathcal{T}_\mathcal{S}$ unique, if it exists.

Proposition 4.7. *For any collection of subsets $\mathcal{S}$, the topology $\mathcal{T}_\mathcal{S}$ exists.*

Proof. The topology

$$\mathcal{T}_\mathcal{S} = \bigcap_{\substack{\text{topologies } \mathcal{T} \\ \text{such that } \mathcal{S} \subseteq \mathcal{T}}} \mathcal{T}$$

has the required properties. □

The following proposition provides an explicit description of $\mathcal{T}_\mathcal{S}$.

Proposition 4.8. *The topology generated by* $\mathcal{S}$ *is*

$$\mathcal{T}_{\mathcal{S}} = \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}$$

i.e. the topology for which $\mathcal{S}$ *is a subbasis.*

Proof. Homework 1 Problem 10. □