THE DUAL WHITEHEAD THEOREMS

J. P. May
The University of Chicago

For Peter Hilton on his 60^{th} birthday

Eckmann-Hilton duality has been around for quite some time and is something we now all take for granted. Nevertheless, it is a guiding principle to "the homotopical foundations of algebraic topology" that is still seldom exploited as thoroughly as it ought to be. In 1971, I noticed that the two theorems commonly referred to as Whitehead's theorem are in fact best viewed as dual to one another. I've never published the details. (They were to appear in a book whose title is in quotes above and which I contracted to deliver to the publishers in 1974; 1984, perhaps?) This seems a splendid occasion to advertise the ideas. The reader is referred to Hilton's own paper [2] for a historical survey and bibliography of Eckmann-Hilton duality. We shall take up where he left off.

The theorems in question read as follows.

<u>Theorem A.</u> A weak homotopy equivalence $e:Y \to Z$ between CW complexes is a homotopy equivalence.

<u>Theorem B.</u> An integral homology isomorphism $e:Y \to Z$ between simple spaces is a weak homotopy equivalence.

In both, we may as well assume that Y and Z are based and (path) connected and that e is a based map. The hypothesis of Theorem A (and conclusion of Theorem B) asserts that $e_*:\pi_*(Y) \to \pi_*(Z)$ is an isomorphism. The hypothesis of Theorem B asserts that $e_*:H_*(X) \to H_*(Y)$ is an isomorphism. A simple space is one whose fundamental group is Abelian and acts trivially on the higher homotopy groups. Theorem B remains true for nilpotent spaces, for which the fundamental group is nilpotent and acts nilpotently on the higher homotopy groups. More general versions have also been proven.

It is well understood that Theorem A is elementary. However, the currently fashionable proof of Theorem B and its generalizations depends on use of the Serre spectral sequence. We shall obtain a considerable generalization of Theorem B by a strict word for word dualization of the simplest possible proof of Theorem A, and our arguments will also yield a generalized form of Theorem A.

We shall work in the good category $\mathcal{J}$ of compactly generated weak Hausdorff based spaces. Essentially the same arguments can be carried out in other good topological categories, for example, in good categories of G-spaces, or spectra, or G-spectra. An axiomatic setting could be developed but would probably obscure the simplicity of the ideas.

We shall use very little beyond fibre and cofibre sequences. Let $X \wedge Y$ be the smash product $X \times Y / X \vee Y$ and let $F(X,Y)$ be the function space of based maps $X \to Y$. The source of duality is the adjunction homeomorphism

(1) $$F(X \wedge Y, Z) \cong F(X, F(Y,Z)).$$

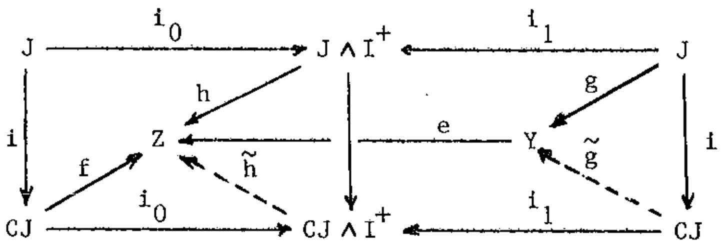
Let $CX = X \wedge I$, $\Sigma X = X \wedge S^1$, $PX = F(I,X)$, and $\Omega X = F(S^1,X)$, where I has basepoint 1 in forming CX and 0 in forming PX. For a based map $f: X \to Y$, let $Cf = Y \cup_f CX$ be the cofibre of f and let $Ff = X \times_f PY$ be the fibre of f. Let $\pi(X,Y)$ denote the pointed set of homotopy classes of based maps $X \to Y$. For spaces J and K, we have the long exact sequences of pointed sets (and further structure as usual)

(2) $$\cdots \longrightarrow \pi(\Sigma^n Cf, K) \longrightarrow \pi(\Sigma^n Y, K) \longrightarrow \pi(\Sigma^n X, K) \longrightarrow \pi(\Sigma^{n-1} Cf, K) \longrightarrow \cdots ;$$

(3) $$\cdots \longrightarrow \pi(J, \Omega^n Ff) \longrightarrow \pi(J, \Omega^n X) \longrightarrow \pi(J, \Omega^n Y) \longrightarrow \pi(J, \Omega^{n-1} Ff) \longrightarrow \cdots .$$

The crux of Theorem A is the following triviality; we shall give the proof since nothing else requires any work.

Lemma 1. Let $e: Y \to Z$ be a map such that $\pi(J, Fe) = 0$. If $hi_1 = eg$ and $hi_0 = fi$ in the following diagram, where i_0, i_1, and i are the evident inclusions, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.



Proof. Define $k_0: J \to Fe$ by $k_0(j) = (g(j), \omega_0(j))$, where $\omega_0(J) \in PZ$ is specified by

$$\omega_0(j)(s) = \begin{cases} f(j,\ 1-2s) & \text{if } s \le 1/2 \\ h(j,\ 2s-1) & \text{if } s \ge 1/2\ . \end{cases}$$

Choose a homotopy $k: J \wedge I^+ \to Fe$ from k_0 to the trivial map and define $\tilde{g}: CJ \to Y$ and $\omega: J \wedge I^+ \to PZ$ by

$$k(j,t) = (\tilde{g}(j,t), \omega(j,t)).$$

Define $\tilde{h}: CJ \wedge I^+ \to Z$ by

$$\tilde{h}(j,s,t) = \omega(j, u(s,t))(v(s,t)),$$

where $u(s,t) = \min(s, 2t)$ and $v(s,t) = \max(\frac{1}{2}(1+t)(1-s), 2t-1)$. Then $\tilde{g}$ and $\tilde{h}$ make the diagram commute.

We now introduce a general version of cellular theory.

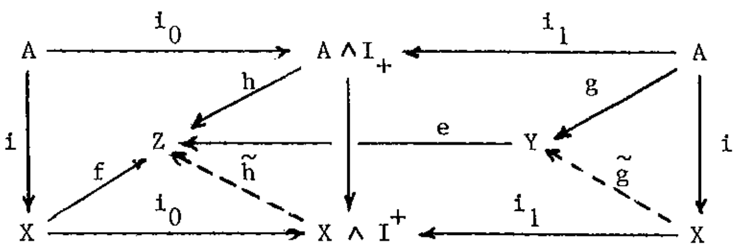
Definition 2. Let $\mathcal{J}$ be any collection of spaces such that $\Sigma J \in \mathcal{J}$ if $J \in \mathcal{J}$. A map $e: Y \to Z$ is said to be a weak $\mathcal{J}$-equivalence if $e_*: \pi(J,Y) \to \pi(J,Z)$ is a bijection for all $J \in \mathcal{J}$. A $\mathcal{J}$-complex is a space X together with subspaces X_n and maps $j_n: J_n \to X_n$, $n \ge 0$, such that $X_0 = \{*\}$, J_n is a wedge of spaces in $\mathcal{J}$, $X_{n+1} = Cj_n$, and X is the union of the X_n. The evident map from the cone on a wedge summand of J_{n-1} into X is called an n-cell. The restriction of j_n to a wedge summand is called an attaching map. A subspace A of a $\mathcal{G}$-complex X is said to be a subcomplex if A is a $\mathcal{G}$-complex such that $A_n \subset X_n$ and the composite of each n-cell $CJ \to A_n \subset A$ and the inclusion $i: A \to X$ is an n-cell of X.

Example 3. Consider $\mathcal{G} = \{ S^n \mid n \geq 0\}$, where we take $S^n = \Sigma S^{n-1}$. A weak $\mathcal{G}$-equivalence between connected spaces is the same thing as a weak homotopy equivalence. We call a $\mathcal{G}$-complex X a cell complex. If J_n is a wedge of n-spheres, then X is a CW-complex with a single vertex and based attaching maps. It is easily verified that any connected CW complex is homotopy equivalent to one of this form. In general, the cells of cell complexes need not be attached only to cells of lower dimension.

Other examples are of interest. For instance, $\mathcal{G}$ might be the set of T-local spheres $\{\Sigma^n S^1_T \mid n \geq 0\}$, where T is a set of primes. In this case, $\mathcal{G}$-complexes lead to the appropriate theory of simply connected T-local CW-complexes.

The acronym (due to Boardman) in the following theorem stands for "homotopy extension and lifting property".

Theorem 4 (HELP). Let A be a subcomplex of a $\mathcal{G}$-complex X and let $e: Y \to Z$ be a weak $\mathcal{G}$-equivalence. If $hi_1 = eg$ and $hi_0 = fi$ in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.



Proof. By (1) and (3) and the fact that $\mathcal{G}$ is closed under suspension, the hypothesis implies that $\pi(J, Fe) = 0$ for all $J \in \mathcal{G}$. We construct compatible maps $\tilde{g}_n: X_n \to Y$ and homotopies $\tilde{h}_n: X_n \wedge I^+ \to Z$ from $f|X_n$ to $e\tilde{g}_n$ by induction on n, starting with the trivial maps $\tilde{g}_0$ and $\tilde{h}_0$ and extending given maps $\tilde{g}_{n-1}$ and $\tilde{h}_{n-1}$ over cells in A_n by use of the given maps g and h and over cells of X_n not in A_n by use of the case (CJ, J) already handled in Lemma 1.

In particular, taking e to be the identity map of Y, we see that the inclusion $i:A \to X$ is a cofibration.

<u>Theorem 5.</u> For every weak $\mathcal{J}$-equivalence $e:Y \to Z$ and every $\mathcal{J}$-complex X, $e_*:\pi(X,Y) \to \pi(X,Z)$ is a bijection.

<u>Proof.</u> We see that e_* is a surjection by application of HELP to the pair $(X,*)$. It is easy to check that $X \wedge I^+$ is a $\mathcal{J}$-complex which contains $X \wedge (\partial I)^+$ as a subcomplex, and we see that e_* is an injection by application of HELP to this pair.

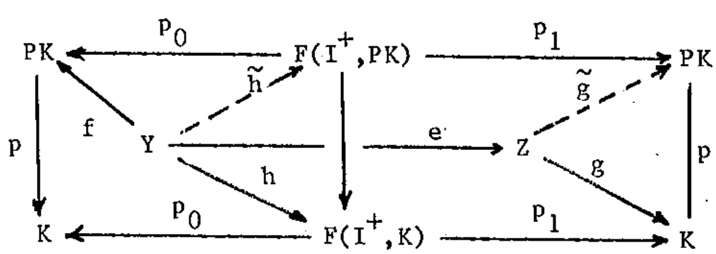
The cellular Whitehead theorem is a formal consequence.

<u>Theorem 6 (Whitehead).</u> Every weak $\mathcal{J}$-equivalence between $\mathcal{J}$-complexes is a homotopy equivalence.

By Example 2, Theorem A is an obvious special case.

Now the fun begins. We dualize everything in sight. The dual of Lemma 1 admits a dual proof which is left as an exercise.

<u>Lemma 1^*.</u> Let $e:Y \to Z$ be a map such that $\pi(Ce,K) = 0$. If $p_1h = ge$ and $p_0h = pf$ in the following diagram, where p_0, p_1, and p are the evident projections, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.



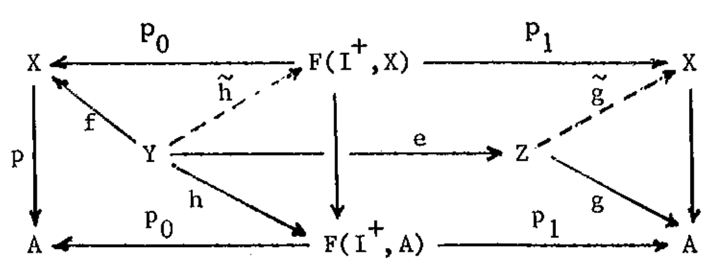
We next introduce the dual "cocellular theory."

<u>Definition 2^*.</u> Let $\mathcal{K}$ be any collection of spaces such that $\Omega K \in \mathcal{K}$ if $K \in \mathcal{K}$. A map $e:Y \to Z$ is said to be a weak $\mathcal{K}$-equivalence if $e^*:\pi(Z,K) \to \pi(Y,K)$ is a bijection for all $K \in \mathcal{K}$. A $\mathcal{K}$-tower is a space X together with maps $X \to X_n$ and $k_n:X_n \to K_n$, $n \geq 0$, such that $X_0 = \{*\}$, K_n

is a product of spaces in $\mathcal{K}$, $X_{n+1} = Fk_n$, and X is the inverse limit of the X_n (via the given maps). The evident map from X to the paths on a factor of K_{n-1} is called an n-cocell. The projection of k_n to a factor is called a coattaching map. A map $p: X \to A$ is said to be a projection onto a quotient tower if A is a $\mathcal{K}$-tower, p is the inverse limit of maps $X_n \to A_n$, and the composite of p and each n-cocell $A \to A_n \to PK$ is an n-cocell of X.

Example 3^*. Let $\mathcal{A}$ be any collection of Abelian groups which contains {0}, for example the collection $\mathcal{A}b$ of all Abelian groups. Let $\mathcal{K}\mathcal{A}$ be the collection of all Eilenberg-MacLane spaces K(A,n) such that $A \in \mathcal{A}$ and $n \geq 0$. (We require Eilenberg-MacLane spaces to have the homotopy types of CW-complexes; this doesn't effect closure under loops by a theorem of Milnor.) A $\mathcal{K}\mathcal{A}b$-tower X such that K_n is a $K(\pi_{n+1}, n+2)$ for $n \geq 0$ is called a simple Postnikov tower and satisfies $\pi_n(X) = \pi_n$. Its coattaching map k_n is usually written k^{n+2} and called a k-invariant.

Theorem 4^* (coHELP). Let A be a quotient tower of a $\mathcal{K}$-tower X and let $e: Y \to Z$ be a weak $\mathcal{K}$-equivalence. If $p_1 h = ge$ and $p_0 h = pf$ in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.



Proof. By (1) and (2) and the fact that $\mathcal{K}$ is closed under loops, the hypothesis implies that $\pi(Ce,K) = 0$ for all $K \in \mathcal{K}$. The conclusion follows inductively by a cocell by cocell application of Lemma 1^*.

In particular, the projection $p: X \to A$ is a fibration.

Theorem 5^*. For every weak $\mathcal{K}$-equivalence $e: Y \to Z$ and every $\mathcal{K}$-tower X, $e^*: \pi(Z,X) \to \pi(Y,X)$ is a bijection.

Proof. The surjectivity and injectivity of e^* result by application of coHELP to the quotient towers $X \to *$ and $F(I^+,X) \to F(\partial I^+,X)$, respectively.

The cocellular Whitehead theorem is a formal consequence.

Theorem 6^*(Whitehead). Every weak $\mathcal{K}$-equivalence between $\mathcal{K}$-towers is a homotopy equivalence.

To derive useful conclusions from these theorems we have to use approximations of spaces by CW-complexes and by Postnikov towers. For a space X of the homotopy type of a CW-complex, we have

$$\tilde{H}^n(X;A) = \pi(X,K(A,n)).$$

However, $\mathcal{K}$-towers hardly ever have the homotopy types of CW-complexes. The best conceptual way around this is to pass from the homotopy category $h\mathcal{J}$ to the category $\bar{h}\mathcal{J}$ obtained from it by inverting its weak homotopy equivalences. For any space X, there is a CW-complex ΓX and a weak homotopy equivalence $\gamma:\Gamma X \to X$. The morphisms of $\bar{h}\mathcal{J}$ can be specified by

$$[X,Y] = \pi(\Gamma X,\Gamma Y),$$

with the evident composition. By Theorem 5, we have $[X,Y] = \pi(X,Y)$ if X has the homotopy type of a CW-complex. Either as a matter of definition or as a consequence of the fact that cohomology is an invariant of weak homotopy type, we have

$$\tilde{H}^n(X;A) = [X,K(A,n)]$$

for any space X.

Now return to Example 3^*. Say that a map $e:Y \to Z$ is an $\mathcal{A}$-cohomology isomorphism if $e^*:H^*(Z;A) \to H^*(Y;A)$ is an isomorphism for all $A \in \mathcal{A}$. If Y and Z are CW-complexes, then e is an $\mathcal{A}$-cohomology isomorphism if and only if it is a weak $\mathcal{K}\mathcal{A}$-equivalence.

Theorem $5^{\#}$. For every $\mathcal{A}$-cohomology isomorphism $e:Y \to Z$ and every $\mathcal{K}\mathcal{A}$-tower X, $e^*:[Z,X] \to [Y,X]$ is a bijection.

Proof. We may as well assume that Y and Z are CW-complexes, and the result is then a special case of Theorem 5^*.

This leads to the cohomological Whitehead Theorem.

Theorem $6^{\#}$ (Whitehead). The following statements are equivalent for a map $e:Y \to Z$ in $\overline{h\mathcal{J}}$ between connected spaces Y and Z of the weak homotopy type of $\mathcal{K}\mathcal{A}$-towers.

(1) e is an isomorphism in $\overline{h\mathcal{J}}$.

(2) $e_*:\pi_*(Y) \to \pi_*(Z)$ is an isomorphism.

(3) $e^*:H^*(Z;A) \to H^*(Y;A)$ is an isomorphism for all $A \in \mathcal{A}$.

(4) $e^*:[Z,X] \to [Y,X]$ is a bijection for all $\mathcal{K}\mathcal{A}$-towers X.

If $\mathcal{A}$ is the collection of modules over a commutative ring R, then the following statement can be added to the list.

(5) $e_*:H_*(Y;R) \to H_*(Z;R)$ is an isomorphism.

Proof. The previous theorem gives (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (1) is formal, and (1) $\Longleftrightarrow$ (2) by the definition of $\overline{h\mathcal{J}}$; (2) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (5) since homology and cohomology are invariants of weak homotopy type, and (5) $\Longrightarrow$ (3) by the universal coefficients spectral sequence.

When $\mathcal{A} = \mathcal{A}b$, the implication (5) $\Longrightarrow$ (2) is the promised generalization of Theorem B. It is almost too general. Given a space X, it is hard to tell whether or not X has the weak homotopy type of a $\mathcal{K}\mathcal{A}b$-tower. If X is simple, or nilpotent, the standard theory of Postnikov towers shows that X does admit such an approximation. However, in Definition 2^* with $\mathcal{K} = \mathcal{K}\mathcal{A}b$, each K_n can be an arbitrary infinite product of K(A,q)'s for varying q and the maps $k_n:X_n \to K_n$ are completely unrestricted. Thus $\mathcal{K}\mathcal{A}b$-towers are a great deal more general than nilpotent Postnikov towers; compare Dror [1].

The applicability of Theorem 6^* to general collections $\mathcal{A}$ is of considerable practical value. A space X is said to be $\mathcal{A}$-complete if $e^*:[Z,X] \to [Y,X]$ is a bijection for all $\mathcal{A}$-cohomology isomorphisms $e:Y \to Z$. Thus Theorem $5^{\#}$ asserts that $\mathcal{K}\mathcal{A}$-towers are $\mathcal{A}$-complete. The completion of a space X at $\mathcal{A}$ is an $\mathcal{A}$-cohomology isomorphism from X to an $\mathcal{A}$-complete space. For a set of primes T, the completion of X at the collection of T-local or T-complete Abelian groups is the localization or completion of X at T; in the latter case, we may equally well use the collection of those Abelian groups which are vector spaces over Z/pZ for some prime $p \in T$.

These ideas give the starting point for an elementary homotopical account of the theory of localizations and completions in which the latter presents little more difficulty than the former; compare Hilton, Mislin, and Roitberg [3]. Details should appear eventually in "The homotopical foundations of algebraic topology". The equivariant generalization of the basic constructions and characterizations has already been published [4,5], and there the present focus on cohomology rather than homology plays a mathematically essential role.

Bibliography

1. E. Dror. A generalization of the Whitehead theorem. Lecture Notes in Mathematics Vol. 249. Springer (1971), 13-22.

2. P. Hilton. Duality in homotopy theory: a rectrospective essay. J. Pure and Applied Algebra 19(1980), 159-169.

3. P. Hilton, G. Mislin, and J. Roitberg. Localization of nilpotent groups and spaces. North-Holland Math. Studies Vol. 15. North Holland. 1975.

4. J. P. May, J. McClure, and G. Triantafillou. Equivariant localization. Bull. London Math. Soc. 14(1982), 223-230.

5. J. P. May. Equivariant completion. Bull. London Math. Soc. 14(1982), 231-237.