Math 527 - Homotopy Theory
Spring 2013
Homework 8, Lecture 3/6

Problem 3. (Whitehead products) For each $n \geq 1$, consider the sphere S^n with its CW-structure having one 0-cell and one n-cell. For any positive integers $p, q \geq 1$, the product $S^p \times S^q$ inherits a CW-structure with four cells, in dimensions 0, p, q, and $p+q$ respectively. The $(p+q-1)$-skeleton of $S^p \times S^q$ is $S^p \vee S^q$ so that the attaching map of the top cell has the form

$$w\colon S^{p+q-1} \to S^p \vee S^q.$$

For any pointed space X, precomposition by w defines an operation

$$\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$$

called the **Whitehead product**, denoted by brackets $[\alpha, \beta] \in \pi_{p+q-1}(X)$.

a. For $p = q = 1$, the Whitehead product takes the form $\pi_1(X) \times \pi_1(X) \to \pi_1(X)$. What is this map?

b. More generally, for $p = 1$ and $q \geq 1$, describe the Whitehead product $\pi_1(X) \times \pi_q(X) \to \pi_q(X)$.

c. Show that a path-connected H-space (c.f. Homework 3 Problem 1) has trivial Whitehead products.

Remark. There are annoying sign conventions involved in answering parts (a) and (b) carefully.