Math 527 - Homotopy Theory
Spring 2013
Homework 1, Lecture 1/16

The fundamental group, revisited.

Problem 2. Let $\mathcal{C}$ be a locally small category with finite products, including a terminal object. Let G be a group object in $\mathcal{C}$. Show that for any object X of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, G)$ is naturally a group.

In other words, the structure maps of G induce a group structure on $\mathrm{Hom}_{\mathcal{C}}(X, G)$, and this assignment

$$\mathrm{Hom}_{\mathcal{C}}(-, G) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Gp}$$

is a functor.

Problem 3. Consider S^1 as the unit circle in $\mathbb{R}^2$ with basepoint $(1, 0)$, and consider the "pinch" map

$$p \colon S^1 \to S^1/S^0 \cong S^1 \vee S^1$$

which collapses the equator $S^0 \subset S^1$, i.e. identifies the points $(1, 0)$ and $(-1, 0)$.

a. Show that the pinch map is (pointed) homotopy coassociative. More precisely, the diagram

$$\begin{array}{ccc}
S^1 & \xrightarrow{\ p\ } & S^1 \vee S^1 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p \vee \mathrm{id}} \\
S^1 \vee S^1 & \xrightarrow[\mathrm{id} \vee p]{} & S^1 \vee S^1 \vee S^1
\end{array}$$

commutes up to pointed homotopy.

In fact, a very similar argument shows that S^1 is a homotopy cogroup object in $\mathbf{Top}_*$. (Do not show this.) Comultiplication is the pinch map $S^1 \to S^1 \vee S^1$, the counit is the constant map $S^1 \to *$, and the coinverse $S^1 \to S^1$ reverses the last component (viewed in $\mathbb{R}^2$).

b. Conclude that for any pointed space (X, x_0), the set $\pi_1(X, x_0)$ is naturally a group.

More precisely, the structure maps of S^1 as homotopy cogroup object induce a group structure on $\pi_1(X, x_0)$, and moreover this assignment defines a (covariant) functor $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gp}$.

Remark. This construction makes transparent the fact that π_1 is a homotopy functor, i.e. it factors through the quotient functor $\mathbf{Top}_* \to \mathrm{Ho}(\mathbf{Top}_*)$.