Math 527 - Homotopy Theory Spring 2013 Homework 12, Lecture 4/10

Problem 2. Let (X, e) be a pointed space. The **James construction** on X is the pointed space obtained by taking words in the elements of X and declaring that e is a unit. Formally, it is the quotient space:

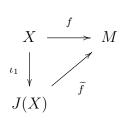
$$J(X):=\coprod_{k\geq 1} X^k/\!\!\sim$$

where \sim is the equivalence relation generated by identifications of the form:

$$(x_1,\ldots,x_{i-1},e,x_{i+1},\ldots,x_k) \sim (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k).$$

a. Show that J(X) is a topological monoid (under concatenation of words).

b. Let M be a topological monoid and $f: X \to M$ a pointed map. Show that there is a unique continuous map of monoids $\tilde{f}: J(X) \to M$ making the diagram



commute. Here $\iota_1 \colon X \to J(X)$ denotes the canonical "inclusion of single-letter words", i.e. the composite

$$X = X^1 \hookrightarrow \prod_{k \ge 1} X^k \twoheadrightarrow J(X).$$

Upshot. This shows that J(X) is in fact the free topological monoid on X. In other words, let $U: \text{TopMon} \to \text{Top}_*$ denote the forgetful functor from topological monoids to pointed spaces. Then the functor $J: \text{Top}_* \to \text{TopMon}$ is left adjoint to U, and $\iota_1: X \to J(X)$ is the unit map of the adjunction.