Math 527 - Homotopy Theory
Spring 2013
Homework 12, Lecture 4/8

Problem 1. Consider the standard inclusions $\mathbb{C}^0 \to \mathbb{C}^1 \to \ldots \to \mathbb{C}^n \to \mathbb{C}^{n+1} \to \ldots$ given by appending a zero in the last coordinate:

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ 0 \end{bmatrix}.$$

These give rise to inclusions $\ldots \to U(n) \to U(n+1) \to \ldots$ described in terms of matrices by:

$$M \mapsto \left[\begin{array}{ccc|c} & & & 0 \\ & M & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where $U(n)$ denotes the Lie group of $n \times n$ unitary matrices with complex coefficients.

a. Show that the connectivity of the map $U(n) \to U(n+1)$ goes to infinity as n goes to infinity.

b. Denote the infinite union $U := \operatorname{colim}_n U(n)$. Deduce from (a) that its homotopy groups satisfy

$$\pi_k U \cong \operatorname*{colim}_n \pi_k U(n)$$

and find n large enough (as a function of k) to guarantee that the map $U(n) \to U$ induces an isomorphism $\pi_k U(n) \xrightarrow{\cong} \pi_k U$.

c. Compute $\pi_k U$ for $0 \leq k \leq 3$.