Math 527 - Homotopy Theory
Hurewicz theorem

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1 Background material

Proposition 1.1. *For all $n \geq 1$, we have $\pi_n(S^n) \cong \mathbb{Z}$, generated by the class of the identity map* $\mathrm{id}\colon S^n \to S^n$.

Proof. The long exact sequence in homotopy of the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$ yields the isomorphism $\pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1)$. The Freudenthal suspension theorem guarantees that $\pi_2(S^2)$ is already stable, so that we have isomorphisms

$$\pi_2(S^2) \underset{\cong}{\xrightarrow{\Sigma}} \pi_3(S^3) \underset{\cong}{\xrightarrow{\Sigma}} \pi_4(S^4) \underset{\cong}{\xrightarrow{\Sigma}} \cdots.$$

Moreover, the suspension map

$$\mathbb{Z} \cong \pi_1(S^1) \xrightarrow{\Sigma} \pi_2(S^2) \cong \mathbb{Z}$$

is surjective, and thus an isomorphism. To conclude, note that class $[\mathrm{id}_{S^1}] \in \pi_1(S^1)$ is a generator, and the suspension map sends the identity to the identity:

$$\Sigma([\mathrm{id}_{S^{n-1}}]) = [\mathrm{id}_{S^n}] \in \pi_n(S^n).$$

□

Alternate proof. Using a bit of differential topology (or a more geometric construction along the lines of Hatcher § 4.1 Exercise 15), consider the degree of a smooth map $f\colon S^n \to S^n$. Since every homotopy class $[f]$ contains a smooth representative, and all such maps have the same degree (i.e. degree is a homotopy invariant), this defines a function

$$\deg\colon \pi_n(S^n) \to \mathbb{Z}.$$

One readily shows that deg is a group homomorphism. One can show moreover that two maps $S^n \to S^n$ with the same degree are homotopic, i.e. deg is injective. The equality $\deg([\mathrm{id}]) = 1$ shows that deg is surjective, hence an isomorphism. □

Remark 1.2. One can show that the definition of degree in differential topology coincides with the following homological definition. The degree of a map $f\colon S^n \to S^n$ is the (well-defined) integer such that the map induced on homology

$$f_*\colon H_n(S^n) \to H_n(S^n)$$

is multiplication by $\deg(f)$, noting the fact $H_n(S^n) \simeq \mathbb{Z}$. In other words, if $u \in H_n(S^n)$ is a generator, then we have $f_*(u) = \deg(f)u \in H_n(S^n)$.

Lemma 1.3. *For $n \geq 2$, we have*

$$\pi_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z},$$

the free abelian group generated by the two summand inclusions $\iota_j\colon S^n \hookrightarrow S^n \vee S^n$ (with $j = 1, 2$).

For $n = 1$, we have

$$\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z},$$

the free group generated by the two summand inclusions $\iota_j\colon S^1 \hookrightarrow S^1 \vee S^1$.

Proof. The case $n = 1$ follows from the Van Kampen theorem. Now assume $n \geq 2$.

Since S^n is $(n-1)$-connected, the inclusion $S^n \vee S^n \to S^n \times S^n$ is $n+n-1 = 2n-1$ connected, and in particular an isomorphism on π_k for $k \leq 2n-2 = n+(n-2)$. We obtain the isomorphism

$$\begin{aligned}\pi_n(S^n \vee S^n) &\cong \pi_n(S^n \times S^n)\\ &\cong \pi_n(S^n) \times \pi_n(S^n)\\ &\cong \pi_n(S^n) \oplus \pi_n(S^n)\\ &\cong \mathbb{Z} \oplus \mathbb{Z},\end{aligned}$$

The generators $[\mathrm{id}_j] \in \pi_n(S^n)$ on the right-hand side correspond to summand inclusions $\iota_j\colon S^n \hookrightarrow S^n \vee S^n$ on the left-hand side. □

Proposition 1.4. *Let J be a set. For $n \geq 2$, we have*

$$\pi_n(\bigvee_{j\in J} S^n) \cong \bigoplus_{j\in J} \mathbb{Z},$$

the free abelian group generated by the summand inclusions $\iota_j\colon S^n \hookrightarrow \bigvee_{j\in J} S^n$.

For $n = 1$, we have

$$\pi_1(\bigvee_{j\in J} S^1) \cong *_{j\in J}\mathbb{Z},$$

the free group generated by the summand inclusions $\iota_j\colon S^1 \hookrightarrow \bigvee_{j\in J} S^1$.

Proof. The case where J is finite follows by applying the same argument as in 1.3 inductively.

For an arbitrary set J, note that a compact subspace of $\bigvee_{j\in J} S^n$ lives in a finite subwedge $\bigvee_{j\in J_\alpha} S^n$, for some finite subset $J_\alpha \subseteq J$. Therefore we obtain a (filtered) colimit

$$\pi_n(\bigvee_{j\in J} S^n) \cong \operatorname*{colim}_{\alpha} \pi_n(\bigvee_{j\in J_\alpha} S^n)$$

where J_α runs over all finite subsets of J (c.f. Homework 6 Problem 3). Said colimit is as claimed in the statement: free abelian group when $n \geq 2$ and free group when $n = 1$. $\square$

2 Weak equivalence implies homology isomorphism

Proposition 2.1. *Let $f\colon X \to Y$ be an n-connected map for some $n \geq 0$. Then f induces an isomorphism on integral homology $f_*\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ for $i < n$ and a surjection when $i = n$.*

In particuliar, any weak homotopy equivalence induces an isomorphism on integral homology $H_*(X;\mathbb{Z}) \xrightarrow{\simeq} H_*(Y;\mathbb{Z})$ (and thus on homology and cohomology with any coefficients, by the universal coefficient theorem).

Direct proof. The case $n = 0$ is clear, since $H_0(X)$ is the free abelian group on $\pi_0(X)$. Now we assume $n \geq 1$.

WLOG X and Y are path-connected. To prove this, note that the natural transformation $\coprod_{C \in \pi_0(X)} C \to X$ is a weak homotopy equivalence and induces an isomorphism on homology.

WLOG f is an embedding, replacing Y by the mapping cylinder $M(f)$ if needed.

By the long exact sequence in homotopy groups of the pair (Y, X), the fact that $f\colon X \to Y$ is n-connected is equivalent to the vanishing of relative homotopy groups $\pi_k(Y, X) = 0$ for $k \leq n$.

By the long exact sequence in homology groups of the pair (Y, X), the desired conclusion on f is equivalent to the vanishing of relative homology groups $H_k(Y, X) = 0$ for $k \leq n$.

Let $\alpha \in H_k(Y, X)$. Then by gluing k-simplices appropriately, one can realize α as coming from a k-dimensional CW-complex K, with a $(k-1)$-dimensional subcomplex $L \subset K$ (realizing the boundary) which is sent to X. See Hatcher Proposition 4.21 for details. In other words, there is a map

$$\sigma\colon (K, L) \to (Y, X)$$

and a class $\overline{\alpha} \in H_k(K, L)$ satisfying $\sigma_*(\overline{\alpha}) = \alpha \in H_k(Y, X)$. The condition $\pi_k(Y, X) = 0$ along with the compression lemma guarantees that σ is homotopic rel L to a map $\sigma'\colon K \to Y$ landing entirely in X. Thus $\sigma_* = \sigma'_*\colon H_k(K, L) \to H_k(Y, X)$ is zero, as it factors through $H_k(X, X) = 0$. □

Using CW-approximation. We first show that CW-approximation induces an isomorphism on homology. Let $\mathrm{Sing}(X)$ denote the singular set of X (which is a simplicial set) and $|\mathrm{Sing}(X)|$ its geometric realization. One can show that the natural map $\epsilon\colon |\mathrm{Sing}(X)| \to X$ is a weak homotopy equivalence. Moreover, $|\mathrm{Sing}(X)|$ admits a CW-structure with a k-cell for each k-simplex in $\mathrm{Sing}(X)$, in which the cellular chain complex of $|\mathrm{Sing}(X)|$ is the chain complex corresponding to the simplicial abelian group obtained by taking the levelwise free abelian group on $\mathrm{Sing}(X)$ – none other than the singular chain complex of X. Thus $|\mathrm{Sing}(X)|$ and X have the same integral homology, and in fact ϵ induces an isomorphism on integral homology.

This shows that the CW-approximation $\epsilon\colon |\mathrm{Sing}(X)| \to X$ induces an isomorphism on homology. But by homotopy uniqueness of CW-approximation, the same conclusion holds for any CW-approximation.

Therefore, we may assume that X and Y are CW-complexes. Indeed, consider the commutative diagram

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ {\scriptstyle\gamma_X}\downarrow\wr & & \wr\downarrow{\scriptstyle\gamma_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where Γ is a functorial CW-approximation. Then f is n-connected if and only Γf is. Since γ_X and γ_Y induce isomorphisms on integral homology, the conclusion about f holds if and only if it holds for Γf.

By the (strong form of the) Whitehead theorem, the induced map

$$f_* \colon [W, X] \to [W, Y]$$

is surjective for any CW-complex W of dimension $d \leq n$ and a bijection for $d < n$. Taking $W = Y_n$ the n-skeleton of Y, the map

$$f_* \colon [Y_n, X] \to [Y_n, Y]$$

is surjective, so that there is a map $g \colon Y_n \to X$ satisfying $[fg] = [\iota_n] \colon Y_n \to Y$, i.e. making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle\iota_n} \\ & & Y_n \end{array}$$

commute up to homotopy, where $\iota_n \colon Y_n \hookrightarrow Y$ is the skeletal inclusion. By cellular homology, $\iota_n \colon Y_n \hookrightarrow Y$ is surjective on homomology H_k for $k \leq n$, and thus so is f.

It remains to prove injectivity on homology H_k for $k < n$. Let $\alpha \in H_k(X)$ be in the kernel of $f_* \colon H_k(X) \to H_k(Y)$, with $k < n$. Since the skeletal inclusion $\iota_{n-1} \colon X_{n-1} \to X$ is surjective on homology H_k, there is a class $\overline{\alpha} \in H_k(X_{n-1})$ satisfying $\iota_{n-1*}(\overline{\alpha}) = \alpha \in H_k(X)$. By cellular approximation, $f \colon X \to Y$ may be assumed cellular, so that its restriction $f|_{X_{n-1}}$ factors through Y_{n-1} (and in particular through Y_n), making the square in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\iota_{n-1}}\uparrow & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle\iota_n} \\ X_{n-1} & \xrightarrow[f|_{X_{n-1}}]{} & Y_n \end{array}$$

commute. Now up to homotopy, we have equality of maps $X_{n-1} \to Y$

$$\begin{aligned} fgf|_{X_{n-1}} &= \iota_n f|_{X_{n-1}} \\ &= f\iota_{n-1} \end{aligned}$$

but recall that the map

$$f_*\colon [X_{n-1}, X] \to [X_{n-1}, Y]$$

is injective, which implies the equality $gf|_{X_{n-1}} = \iota_{n-1}$ up to homotopy. In homology we obtain

$$\begin{aligned} \alpha &= \iota_{n-1*}(\overline{\alpha}) \\ &= g_* f|_{X_{n-1}*}(\overline{\alpha}) \\ &= g_*(0) \\ &= 0. \end{aligned}$$

Indeed, the class $f|_{X_{n-1}*}(\overline{\alpha})$ satisfies

$$\begin{aligned} \iota_{n*} f|_{X_{n-1}*}(\overline{\alpha}) &= f_* \iota_{n-1*}(\overline{\alpha}) \\ &= f_*(\alpha) \\ &= 0 \end{aligned}$$

but again by cellular homology, $\iota_{n*}\colon H_k(Y_n) \to H_k(Y)$ is injective for $k < n$. □

Remark 2.2. If all we care about is the special case $n = \infty$, then no need to play around with skeletal inclusions. By Whitehead, a weak homotopy equivalence between CW-complexes is a homotopy equivalence, and therefore induces an isomorphism on homology.

Corollary 2.3. *Let $f\colon X \to Y$ be an n-connected map for some $n \geq 0$, and let M be an abelian group. Then the following holds.*

1. *The induced map on homology with coefficients in M*

$$f_*\colon H_i(X; M) \to H_i(Y; M)$$

is an isomorphism for $i < n$ and a surjection when $i = n$.

2. *The induced map on cohomology with coefficients in M*

$$f^*\colon H^i(Y; M) \to H^i(X; M)$$

is an isomorphism for $i < n$ and an injection when $i = n$.

Proof. 1. The universal coefficient theorem for homology provides a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} M & \longrightarrow & H_i(X;M) & \longrightarrow & \operatorname{Tor}_1^{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} M & \longrightarrow & H_i(Y;M) & \longrightarrow & \operatorname{Tor}_1^{\mathbb{Z}}(H_{i-1}(Y;\mathbb{Z}), M) & \longrightarrow & 0 \end{array}$$

where the two outer downward maps are isomorphisms when $i < n$, and hence so is the middle downward map $H_i(X;M) \xrightarrow{\cong} H_i(Y;M)$.

In the case $i = n$, the left downward arrow is surjective, while the right downward arrow is an isomorphism. Therefore the middle downward map $H_i(X;M) \twoheadrightarrow H_i(Y;M)$ is surjective.

2. The universal coefficient theorem for cohomology provides a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(H_{i-1}(Y;\mathbb{Z}),M) & \longrightarrow & H^i(Y;M) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_i(Y;\mathbb{Z}),M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}),M) & \longrightarrow & H^i(X;M) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}),M) & \longrightarrow & 0
\end{array}$$

where the two outer downward maps are isomorphisms when $i < n$, and hence so is the middle downward map $H^i(Y;M) \xrightarrow{\cong} H^i(X;M)$.

In the case $i = n$, the left downward arrow is an isomorphism, while the right downward arrow is injective. Therefore the middle downward map $H^i(Y;M) \hookrightarrow H^i(X;M)$ is injective. □

Example 2.4. The map $S^n \to *$ is n-connected. The induced map on homology with coefficients

$$f_* \colon H_i(S^n;M) \to H_i(*;M)$$

is indeed an isomorphism for $i < n$, and the surjection $M \twoheadrightarrow 0$ for $i = n$.

The induced map on cohomology with coefficients

$$f^* \colon H^i(*;M) \to H^i(S^n;M)$$

is indeed an isomorphism for $i < n$ and the injection $0 \hookrightarrow M$ for $i = n$.

3 Hurewicz morphism

Let $n \geq 1$ and recall the homology group $H_n(S^n) \simeq \mathbb{Z}$. There is no canonical choice of generator (between the two choices), so we will fix generators once and for all. More precisely, pick a generator $u_1 \in H_1(S^1)$ and pick the remaining generators $u_n \in H_n(S^n)$ so that via the suspension isomorphism

$$H_{n+1}(S^{n+1}) \cong H_n(S^n)$$

u_{n+1} corresponds to u_n, for all $n \geq 1$.

Remark 3.1. One can (and should) start with $n = 0$, but then one must use reduced homology throughout, so that the condition $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ also holds when $n = 0$. Moreover, the suspension isomorphism for reduced homology $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$ holds for all $n \geq -1$, whereas the suspension isomorphism for unreduced homology $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ only holds for $n \geq 1$.

Definition 3.2. Let $n \geq 1$ and let $\alpha \colon S^n \to X$ be any map. Consider the induced map on integral homology

$$\mathbb{Z} \simeq H_n(S^n) \xrightarrow{H_n(\alpha)} H_n(X)$$

and define $h(\alpha) := H_n(\alpha)(u_n) \in H_n(X)$, the "image of 1" under that map. Since homology is a homotopy functor, this assignment is a well-defined function

$$h \colon \pi_n(X) \to H_n(X)$$

called the **Hurewicz morphism**.

Proposition 3.3. *The Hurewicz map is a group homomorphism.*

Proof. Let $\alpha, \beta \colon S^n \to X$ be two maps. Their sum in $\pi_n(X)$ (where "sum" might be non-commutative when $n = 1$) is represented by the composite

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\nabla} X$$

where $p \colon S^n \to S^n \vee S^n$ is the usual pinch map, and $\nabla \colon X \vee X \to X$ is the fold map. Applying homology and using the natural isomorphism $\widetilde{H}_*(X \vee Y) \cong \widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
H_n(S^n) & \xrightarrow{p_*} & H_n(S^n \vee S^n) & \xrightarrow{(\alpha \vee \beta)_*} & H_n(X \vee X) & \xrightarrow{\nabla_*} & H_n(X) \\
& {\scriptstyle\Delta}\searrow & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & \nearrow{\scriptstyle\nabla} & \\
& & H_n(S^n) \oplus H_n(S^n) & \xrightarrow{\alpha_* \oplus \beta_*} & H_n(X) \oplus H_n(X). & &
\end{array}$$

The image of the generator $u_n \in H_n(S^n)$ along the top composite is $h(\alpha + \beta)$ and along the bottom composite is $h(\alpha) + h(\beta)$. □

Proposition 3.4. *The Hurewicz morphism is natural, and compatible with the suspension map, in the sense that the diagram*

$$\begin{CD} \pi_n(X) @>{h}>> H_n(X) \\ @V{\Sigma}VV @V{\cong}V{\Sigma}V \\ \pi_{n+1}(\Sigma X) @>>{h}> H_{n+1}(\Sigma X) \end{CD}$$

commutes.

Proof. **Naturality.** This follows from functoriality of homology. Let $f\colon X \to Y$ be a map. We want to show that the diagram

$$\begin{CD} \pi_n(X) @>{h}>> H_n(X) \\ @V{\pi_n(f)}VV @VV{H_n(f)}V \\ \pi_n(Y) @>>{h}> H_n(Y) \end{CD}$$

commutes. Given $\alpha \in \pi_n(X)$ represented by a map $\alpha\colon S^n \to X$ we have:

$$\begin{aligned} h\left(\pi_n(f)(\alpha)\right) &= h(f\alpha) \\ &= H_n(f\alpha)(u_n) \\ &= H_n(f)H_n(\alpha)(u_n) \\ &= H_n(f)\left(h(\alpha)\right). \end{aligned}$$

Suspension. This follows from naturality of the suspension map on homology, i.e. commutativity of the diagram

$$\begin{CD} H_n(W) @>{H_n(g)}>> H_n(X) \\ @V{\Sigma}VV @VV{\Sigma}V \\ H_{n+1}(\Sigma W) @>>{H_{n+1}(\Sigma g)}> H_{n+1}(\Sigma X) \end{CD}$$

for any map $g\colon W \to X$.

Given $\alpha \in \pi_n(X)$ represented by a map $\alpha\colon S^n \to X$ we have:

$$\begin{aligned} h\left(\Sigma(\alpha)\right) &= h(\Sigma\alpha) \\ &= H_{n+1}(\Sigma\alpha)(u_{n+1}) \\ &= H_{n+1}(\Sigma\alpha)(\Sigma u_n) \;\text{ by our convention on generators } u_n \\ &= \Sigma H_n(\alpha)(u_n) \\ &= \Sigma\left(h(\alpha)\right). \end{aligned}$$

□

4 Hurewicz theorem

First we treat the case $n = 1$ separately.

Lemma 4.1. *For any wedge of circles, the Hurewicz morphism*

$$h\colon \pi_1(\bigvee_{j\in J} S^1) \to H_1(\bigvee_{j\in J} S^1)$$

is the abelianization morphism.

Proof. By 1.4, the left-hand side is the free group generated by summand inclusions $\iota_j\colon S^1 \hookrightarrow \bigvee_{j\in J} S^1$. The Hurewicz map sends those to classes

$$h(\iota_j) = \iota_{j*}(u_1) \in H_1(\bigvee_{j\in J} S^1) \cong \bigoplus_{j\in J} H_1(S^1).$$

These classes form a basis of the right-hand side as a free abelian group. This explicit description exhibits h as the abelianization. □

Theorem 4.2. *Let X be a path-connected space. Then the Hurewicz morphism for $n = 1$*

$$h\colon \pi_1(X) \to H_1(X)$$

is the abelianization morphism.

Proof. See Hatcher § 2.A Theorem 2A.1 or May § 15.1. □

Lemma 4.3. *Let $n \geq 2$. For any wedge of n-spheres, the Hurewicz morphism*

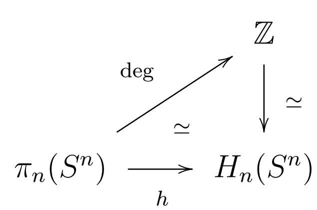
$$h\colon \pi_n(\bigvee_{j\in J} S^n) \to H_n(\bigvee_{j\in J} S^n)$$

is an isomorphism.

Proof. First, note that the statement holds for a single sphere. For $\alpha \in \pi_n(S^n)$, its Hurewicz image is

$$h(\alpha) = H_n(\alpha)(u_n) = \deg(\alpha)u_n \in H_n(S^n).$$

Hence, up to the choice of generator $H_n(S^n) \simeq \mathbb{Z}$, the Hurewicz map is the degree map



which we know is an isomorphism, by 1.1.

Now consider an arbitrary wedge of n-spheres. By 1.4, the left-hand side is the free abelian group generated by summand inclusions $\iota_j \colon S^n \hookrightarrow \bigvee_{j\in J} S^n$. The Hurewicz map sends those to classes

$$h(\iota_j) = \iota_{j*}(u_n) \in H_n(\bigvee_{j\in J} S^n) \cong \bigoplus_{j\in J} H_n(S^n).$$

These classes form a basis of the right-hand side as a free abelian group. □

Theorem 4.4. *Let X be an $(n-1)$-connected space for some $n \geq 2$. Then the Hurewicz morphism*

$$h \colon \pi_n(X) \to H_n(X)$$

is an isomorphism.

Proof. By 2.1, we may replace X by a weakly equivalent space. By CW-approximation, we may assume X is a CW-complex with a single 0-cell and cells in dimension at least n. Since both $\pi_n(X)$ and $H_n(X)$ are determined by the $(n+1)$-skeleton X_{n+1}, we may assume X is $(n+1)$-dimensional. The n-skeleton $X_n = \bigvee_j S^n$ is a wedge of n-sphere, and therefore X is obtained as the cofiber

$$\bigvee_i S^n \xrightarrow{\ \varphi\ } \bigvee_j S^n \xrightarrow{\ g\ } X$$

of a map $\varphi \colon A \to B$ between wedges of n-spheres. (Here we used the fact that for well-pointed spaces, an unpointed cofiber is equivalent to a pointed cofiber. We may assume that all attaching maps in the CW-structure are pointed.)

Applying homology to the cofiber sequence $A \to B \to X$ yields an exact sequence

$$H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A) = 0$$

where we used $H_{n-1}(A) = H_{n-1}(\bigvee_i S^n) \cong \bigoplus_i H_{n-1}(S^n) = 0$.

Now $g \colon B \to X$ is the inclusion of the n-skeleton, and therefore an n-connected map, so that $\pi_n(B) \twoheadrightarrow \pi_n(X)$ is surjective. In other words, the sequence

$$\pi_n(B) \twoheadrightarrow \pi_n(X) \to 0$$

is exact.

Consider the homotopy pushout square

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B \\ \downarrow & \text{ho} \ \ulcorner & \downarrow g \\ * & \longrightarrow & X \end{array}$$

where $A \to *$ is n-connected and the attaching map $\varphi \colon A \to B$ is $(n-1)$-connected. By Blakers-Massey homotopy excision, the square is $n + (n-1) - 1 = 2n-2$ Cartesian. Therefore the map induced on the vertical homotopy fibers $A \to F(g)$ is $(2n-2)$-connected. In particular, since $n \geq 2$, we have $n \leq n + (n-2) = 2n-2$ and so the induced map

$$\pi_n(A) \twoheadrightarrow \pi_n(F(g))$$

is surjective. Therefore applying π_n to the cofiber sequence $A \to B \to X$ yields the sequence

$$\begin{array}{ccccc} & \nearrow & \pi_n(F(g)) & & \\ & & \downarrow \psi_* & & \\ \pi_n(A) & \xrightarrow{\varphi_*} & \pi_n(B) & \xrightarrow{g_*} & \pi_n(X) \end{array}$$

which is exact at $\pi_n(B)$, because of the equality

$$\operatorname{im} \varphi_* = \operatorname{im} \psi_* = \ker g_*.$$

Putting these facts together, applying the Hurewicz morphism to the cofiber sequence $A \to B \to X$ yields a commutative diagram

$$\begin{array}{ccccccc} \pi_n(A) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\ h \downarrow & & h \downarrow & & h \downarrow & & \\ H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & 0 \end{array}$$

where both rows are exact.

By 4.3, the first two downward maps $h\colon \pi_n(A) \xrightarrow{\cong} H_n(A)$ and $h\colon \pi_n(B) \xrightarrow{\cong} H_n(B)$ are isomorphisms. By the 5-lemma, the last map $h\colon \pi_n(X) \xrightarrow{\cong} H_n(X)$ is also an isomorphism. □

Remark 4.5. Proposition 1.1 can be recovered as an easy special case. Since the sphere S^n is $(n-1)$-connected, the Hurewicz map $\pi_n(S^n) \xrightarrow{\cong} H_n(S^n) \simeq \mathbb{Z}$ is an isomorphism.

Corollary 4.6. *Let X be an $(n-1)$-connected space for some $n \geq 2$. Then the integral homology of X satisfies $H_k(X) = 0$ for $k < n$ and $H_n(X) \cong \pi_n(X)$.*

Proof. Apply the Hurewicz theorem successively in dimensions $1, 2, \ldots, n$. □

In other words, for simply-connected spaces, the bottom non-trivial homotopy group coincides with the bottom non-trivial homology group.

5 Relative version

There is a similarly defined relative Hurewicz morphism

$$h\colon \pi_n(X, A) \to H_n(X, A)$$

for $n \geq 2$, using the fact $H_n(D^n, \partial D^n) \cong \widetilde{H}_n(D^n/\partial D^n) \simeq \mathbb{Z}$.

Proposition 5.1. *The relative Hurewicz morphism is natural, and is a group homomorphism. Moreover, it is compatible with the long exact sequences in homotopy and homology of a pair (X, A).*

Theorem 5.2. *Let (X, A) be an $(n-1)$-connected pair for some $n \geq 2$, where A is path-connected (and therefore so is X). Assume moreover that A is simply-connected (and therefore so is X). Then we have $H_i(X, A) = 0$ for $i < n$ and the Hurewicz map $h\colon \pi_n(X, A) \xrightarrow{\cong} H_n(X, A)$ is an isomorphism.*

One can weaken the connectivity assumptions on A and X, but then the correct statement becomes more subtle.

Theorem 5.3. *Let (X, A) be an $(n-1)$-connected pair for some $n \geq 2$, where A is path-connected (and therefore so is X). Then the Hurewicz map $h\colon \pi_n(X, A) \to H_n(X, A)$ is the map factoring out the action of $\pi_1(A)$. More precisely, the quotient by the normal subgroup generated by all elements $\alpha - \gamma \cdot \alpha$, for $\gamma \in \pi_1(A)$ and $\alpha \in \pi_n(X, A)$.*

In particular, if $\pi_n(X, A)$ vanishes, then so does $H_n(X, A)$.

Proof. See tom Dieck § 20.1, in particular Theorem 20.1.11. □

Remark 5.4. Proposition 2.1 can be recovered using this theorem. The assumption was that the relative homotopy groups $\pi_i(Y, X)$ vanish for $i \leq n$. In the case $n \geq 2$, the relative Hurewicz theorem 5.3 implies that the relative homology groups $H_i(Y, X)$ also vanish for $i \leq n$.

To treat the case $n = 1$, use the functorial description $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$. The map $\pi_1(f)\colon \pi_1(X) \twoheadrightarrow \pi_1(Y)$ being surjective guarantees that $H_1(f)\colon H_1(X) \to H_1(Y)$ is also surjective, from which we conclude $H_1(Y, X) = 0$.

6 Homology Whitehead theorem

Consider a map $f\colon X \to Y$. In this section it will be useful to keep the commutative diagram

$$\begin{array}{ccccccccccccccc}
\ldots & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(Y) & \longrightarrow & \pi_2(Y,X) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(Y,X) & \longrightarrow & \pi_0(X) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H_2(X) & \longrightarrow & H_2(Y) & \longrightarrow & H_2(Y,X) & \longrightarrow & H_1(X) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(Y,X) & \longrightarrow & H_0(X)
\end{array}$$

in mind, though we will not explicitly refer to it.

Recall from 2.1 that a weak homotopy equivalence induces isomorphisms on homology and cohomology with any (trivial) coefficients. More is true.

Proposition 6.1. *A weak homotopy equivalence induces isomorphisms on homology and cohomology with any local coefficients.*

Proof. This can be proved by passing to universal covers, and using the fact $H_*(\widetilde{X};\mathbb{Z}) \cong H_*(X;\mathbb{Z}\pi_1(X))$ (Hatcher § 3.H Example 3H.2 or [1, Exercise 73]) along with an appropriate analogue of the universal coefficient theorem. □

Proposition 6.2. *Let X and Y be simply-connected spaces and $f\colon X \to Y$ a map which induces an isomorphism on integral homology $f_*\colon H_*(X;\mathbb{Z}) \xrightarrow{\cong} H_*(Y;\mathbb{Z})$. Then f is a weak homotopy equivalence.*

Proof. We know X and Y are path-connected. Since Y is moreover simply-connected, we have $\pi_1(Y,X) = 0$ so that the pair (Y,X) is 1-connected. By the relative Hurewicz theorem, we have the isomorphism

$$\pi_2(Y,X) \xrightarrow{\cong} H_2(Y,X) = 0$$

where the relative homology group vanishes since f induces isomorphisms on integral homology. Thus the pair (Y,X) is 2-connected. Repeating this argument inductively, we conclude that all relative homotopy groups $\pi_i(Y,X)$ vanish for $i \geq 1$, so that $f\colon X \xrightarrow{\sim} Y$ is a weak homotopy equivalence. □

Proposition 6.3. *Let $f\colon X \to Y$ be a map inducing an equivalence of fundamental groupoids Π_1 and an isomorphism on homology with any local coefficients. Then f is a weak homotopy equivalence.*

Proof. WLOG X and Y are path-connected. Then the condition on fundamental groupoids means that f induces an isomorphism on π_1.

By CW-approximation, we may assume that X and Y are CW-complexes. Indeed, consider the commutative diagram

$$\begin{array}{ccc}
\Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\
{\scriptstyle \gamma_X}\downarrow\wr & & \wr\downarrow{\scriptstyle \gamma_Y} \\
X & \xrightarrow[f]{} & Y
\end{array}$$

where Γ is a functorial CW-approximation. By 2-out-of-3, Γf is a weak homotopy equivalence if and only if f is. By 6.1, both γ_X and γ_Y induce isomorphisms on π_1 and homology with local coefficients. But the class of all such maps also satisfies 2-out-of-3.

All we want here is that X and Y are locally contractible, and in particular locally path-connected and semi-locally simply-connected, and thus admit universal covers. Let $p_X \colon \widetilde{X} \to X$ and $p_Y \colon \widetilde{Y} \to Y$ denote the universal covers of X and Y respectively. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ trivially induces an isomorphism on π_1. Using the isomorphisms

$$H_*(\widetilde{X};\mathbb{Z}) \cong H_*(X;\mathbb{Z}\pi_1(X))$$

$$H_*(\widetilde{Y};\mathbb{Z}) \cong H_*(Y;\mathbb{Z}\pi_1(Y))$$

we deduce that $\widetilde{f}$ induces an isomorphism on integral homology. Indeed, by assumption f induces an isomorphism on homology with local coefficients

$$f_* \colon H_*(X; f^*\mathbb{Z}\pi_1(Y)) \xrightarrow{\simeq} H_*(Y;\mathbb{Z}\pi_1(Y))$$

and since f also induces an isomorphism on π_1, we have the isomorphism

$$H_*(X; f^*\mathbb{Z}\pi_1(Y)) \simeq H_*(X;\mathbb{Z}\pi_1(X)).$$

By 6.2, it follows that $\widetilde{f} \colon \widetilde{X} \xrightarrow{\sim} \widetilde{Y}$ is a weak homotopy equivalence. Since both covering maps $p_X \colon \widetilde{X} \to X$ and $p_Y \colon \widetilde{Y} \to Y$ induce isomorphisms on higher homotopy groups π_i for $i \geq 2$, then so does $f \colon X \to Y$. Therefore f is a weak homotopy equivalence. □

Warning 6.4. It is **NOT** true in general that a map $f \colon X \to Y$ which induces isomorphisms on π_1 and on integral homology $H_*(-;\mathbb{Z})$ is a weak homotopy equivalence. Not even if f induces isomorphisms on an enormous range of lowest homotopy groups. Not even if f is a nice map, e.g. the inclusion of a subcomplex into a finite CW-complex. See for example Homework 10 Problem 1.

That said, there are sufficient conditions to guarantee this kind of conclusion, which will in fact generalize 6.2.

Definition 6.5. A space X is **simple** if for any choice of basepoint $x \in X$, the fundamental group $\pi_1(X,x)$ acts trivially on all homotopy groups $\pi_n(X,x)$ for $n \geq 1$. (In particular, the fundamental group at any basepoint must be abelian.)

Proposition 6.6. *Let X and Y be simple spaces and $f \colon X \to Y$ a map inducing an isomorphism on integral homology $f_* \colon H_*(X;\mathbb{Z}) \xrightarrow{\simeq} H_*(Y;\mathbb{Z})$. Then f is a weak homotopy equivalence. (Note that f automatically induces an isomorphism on $\pi_1 \cong H_1$ in this case, since all fundamental groups involved are abelian.)*

Proof. See [2, Theorem 6#] and the discussion afterwards. Here is a sketch of the proof.

By the universal coefficient theorem, f induces an isomorphism on homology and cohomology with any trivial coefficients $f^*\colon H^*(Y;M) \xrightarrow{\simeq} H^*(X;M)$. Recall that (reduced) cohomology is represented by Eilenberg-MacLane spaces

$$\widetilde{H}^n(X;M) \cong [\Gamma X, K(M,n)]_*$$

for spaces with the homotopy type of a CW-complex (and we need CW-approximation Γ for arbitrary spaces).

Since X and Y are simple, they admit simple Postnikov towers. By an inductive argument, one can show that f induces a bijection on derived homotopy classes of maps

$$f^*\colon [\Gamma Y, Z] \xrightarrow{\simeq} [\Gamma X, Z]$$

whenever Z is an (appropriate) tower built out of Eilenberg-MacLane spaces. This applies in particular to $Z = X$ and $Z = Y$, from which we deduce that $f\colon X \to Y$ is an isomorphism in the weak homotopy category $w\,\mathrm{Ho}(\mathbf{Top})$, i.e. a weak homotopy equivalence. □

Combining these propositions with the Whitehead theorem, we obtain the so-called homology Whitehead theorems.

Corollary 6.7. *Let $f\colon X \to Y$ be a map between CW-complexes. In each of the following cases, it follows that f is a homotopy equivalence.*

1. *f induces an equivalence on fundamental groupoids and an isomorphism on homology with any local coefficients.*
2. *X and Y are simple, and f induces an isomorphism on integral homology.*
3. *(Particular case.) X and Y are simply-connected, and f induces an isomorphism on integral homology.*

Remark 6.8. By the usual 2-out-of-3 argument (c.f. Homework 7 Problem 1a), the statement holds more generally when X and Y have the *homotopy type* of CW-complexes, instead of being CW-complexes.

7 Other applications

Proposition 7.1. *Every simply-connected closed 3-manifold is homotopy equivalent to a sphere* S^3.

Proof. Let X be a a simply-connected closed 3-manifold. We will first check that X is a homology sphere.

Since X is path-connected, we have $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. Since X is moreover simply-connected, we have $H_1(X;\mathbb{Z}) \cong \pi_1(X)_{\mathrm{ab}} = 0$. Also, since X is simply-connected, it is orientable, and we have $H_3(X;\mathbb{Z}) \cong \mathbb{Z}$. By Poincaré duality, we have $H_2(X;\mathbb{Z}) \cong H^1(X;\mathbb{Z})$. By the universal coefficient theorem for cohomology, we have a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_0(X;\mathbb{Z}),\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \to 0$$

which yields $H^1(X;\mathbb{Z}) = 0$ since $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$ is a projective $\mathbb{Z}$-module and $H_1(X;\mathbb{Z}) = 0$.

By the Hurewicz theorem, the Hurewicz map $h\colon \pi_3(X) \xrightarrow{\cong} H_3(X;\mathbb{Z}) \simeq \mathbb{Z}$ is an isomorphism. Pick a generator $\alpha \in \pi_3(X) \simeq \mathbb{Z}$ and represent it by a map $\alpha\colon S^3 \to X$. Then α induces isomorphisms on integral homology $\alpha_*\colon H_*(S^3;\mathbb{Z}) \xrightarrow{\simeq} H_*(X;\mathbb{Z})$. Since S^3 and X are simply-connected and have the homotopy type of CW-complexes, the map $\alpha\colon S^3 \xrightarrow{\simeq} X$ is a homotopy equivalence. □

The argument above proves the following.

Proposition 7.2. *Let X be a simply-connected closed n-manifold, for some $n \geq 2$. If X is a homology sphere (i.e. its integral homology is isomorphic to that of S^n), then X is homotopy equivalent to a sphere S^n.*

Remark 7.3. The assumption that X is simply-connected cannot be dropped in general. There are homology spheres which are not simply-connected, and thus not homotopy equivalent to S^n. See for example the Poincaré homology sphere of dimension 3, described here:

`http://www.map.mpim-bonn.mpg.de/Poincar%C3%A9%E2%80%99s_homology_sphere`

References

[1] James F. Davis and Paul Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2001.

[2] J. P. May, *The dual Whitehead theorems*, Topological topics, London Math. Soc. Lecture Note Ser., vol. 86, Cambridge Univ. Press, Cambridge, 1983, pp. 46–54.