Math 527 - Homotopy Theory
Homotopy pullbacks

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March 8, 2013[*]

The notion of homotopy pullback is meant as a homotopy invariant approximation of strict pullbacks, which are not homotopy invariant.

1 Definitions

Definition 1.1. Consider a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in **Top**. The **homotopy pullback** of the diagram is

$$\begin{aligned} X \times_Z^h Y &:= X \times_Z Z^I \times_Z Y \\ &= \{(x, \gamma, y) \in X \times Z^I \times Y \mid \gamma(0) = f(x), \gamma(1) = g(y)\} \end{aligned}$$

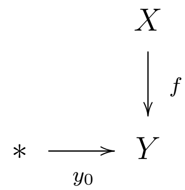
together with the projection maps making the diagram

$$\begin{array}{ccc} X \times_Z^h Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{1.2}$$

commute up to homotopy. In fact, the diagram (1.2) commutes up to a canonical homotopy $H \colon X \times_Z^h Y \to Z$ from fp_X to gp_Y given by $H(x, \gamma, y, t) = \gamma(t)$.

[*]Updated May 8, 2017. I thank David Carchedi for discussions that helped clarify the proof of Proposition 2.3.

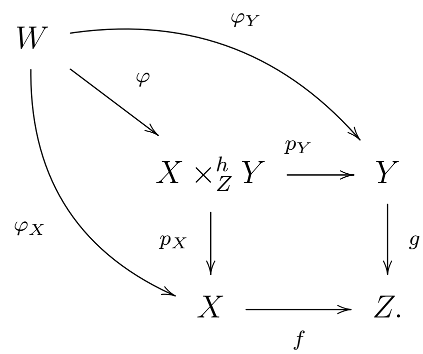
Example 1.3. The homotopy pullback of the diagram



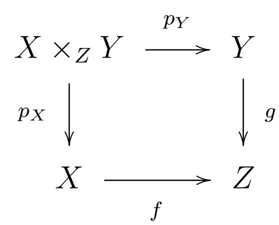
is the homotopy fiber of f over the basepoint $y_0 \in Y$:

$$* \times^h_Y X \cong \{(\gamma, x) \in Y^I \times X \mid \gamma(0) = y_0, \gamma(1) = f(x)\} \cong F(f).$$

A map $\varphi\colon W \to X \times^h_Z Y$ consists of the data of maps $\varphi_X\colon W \to X$ and $\varphi_Y\colon W \to Y$, given by $\varphi_X = p_X\varphi$ and $\varphi_Y = p_Y\varphi$, along with a homotopy from $f\varphi_X$ to $g\varphi_Y$, as illustrated in the diagram:



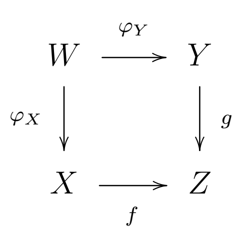
In particular, the strictly commutative diagram



yields a canonical map

$$X \times_Z Y \to X \times^h_Z Y$$

from the strict pullback to the homotopy pullback, which generalizes the inclusion of the strict fiber into the homotopy fiber.
More generally, any strictly commutative diagram



yields a canonical map

$$W \to X \times_Z^h Y \tag{1.4}$$

via the stationary homotopy of $f\varphi_X = g\varphi_Y$, so that the map (1.4) is the composite

$$W \to X \times_Z Y \to X \times_Z^h Y.$$

Another description of the homotopy pullback is the strict pullback of the path space constructions on f and g, as illustrated in the diagram:

$$\begin{array}{ccccc} & & & & Y \\ & & & & \downarrow{\scriptstyle \simeq} \\ & & P(f) \times_Z P(g) & \longrightarrow & P(g) \\ & & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[\simeq]{} & P(f) & \longrightarrow & Z. \end{array}$$

In other words, the homotopy pullback is obtained by replacing the two maps by fibrations and then taking the strict pullback. We will see shortly that it suffices to replace only one of the maps by a fibration.

2 Basic properties

In order to prove Proposition 2.3, we make a short excursion into 2-categories.

Lemma 2.1. *Let $\mathcal{C}$ be a (strict) 2-category, $f\colon X \xrightarrow{\simeq} Y$ an equivalence in $\mathcal{C}$, $g\colon Y \to X$ a 1-morphism, and $\eta\colon 1_X \cong gf$ a 2-isomorphism. Then there exists a unique 2-isomorphism $\epsilon\colon fg \cong 1_Y$ making (f, g, η, ϵ) into an adjoint equivalence.*
Likewise with the roles of η and ϵ reversed.

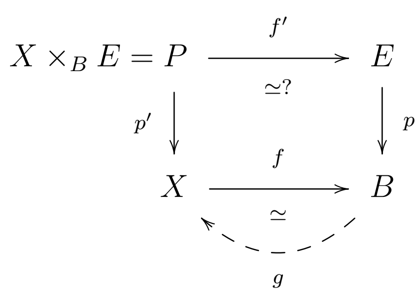
Proof. See [4, §3] or [2, Exercise 2.2]. □

We will apply this lemma in the following 2-category.

Example 2.2. Consider the track category (i.e. groupoid-enriched category) of topological spaces. Objects are topological spaces X, 1-morphisms are continuous maps $f\colon X \to Y$, and 2-morphisms $[H]\colon f \Rightarrow g$ are tracks, i.e., homotopy classes of homotopies $H\colon X \times I \to Y$ from f to g rel $X \times \partial I$.

Proposition 2.3. *The pullback of a homotopy equivalence along a fibration is again a homotopy equivalence.*

Proof. Consider the (strict) pullback diagram



where $p\colon E \to B$ is a fibration and $f\colon X \xrightarrow{\simeq} B$ is a homotopy equivalence. We want to show that $f'\colon P \to E$ is a homotopy equivalence. The proof will proceed in two steps:

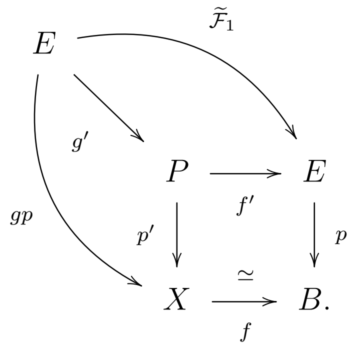
1. Produce a map $g'\colon E \to P$ satisfying $f'g' \simeq 1_E$.

2. Show that g' also satisfies $g'f' \simeq 1_P$, and thus g' is homotopy inverse to f'.

Step 1. Let $g\colon B \to X$ be a homotopy inverse of $f\colon X \xrightarrow{\simeq} B$. Choose a homotopy $F\colon B \times I \to B$ from 1_B to fg; we denote this by $F\colon 1_B \Rightarrow fg$. Precomposing by p yields the homotopy $Fp\colon E \times I \to B$ from p to fgp, that is, $Fp\colon p \Rightarrow fgp$. Note that at time 0, the map $(Fp)_0 = p\colon E \to B$ lifts to E to the map $\widetilde{\mathcal{F}}_0 = 1_E\colon E \to E$.
Since p is a fibration, we can lift the homotopy Fp to a homotopy $\widetilde{\mathcal{F}}\colon E \times I \to E$ with prescribed lift at time 0, $\widetilde{\mathcal{F}}_0 = 1_E$. At time 1, we obtain a map $\widetilde{\mathcal{F}}_1\colon E \to E$ which lifts

$$p\widetilde{\mathcal{F}}_1 = (Fp)_1 = fgp\colon E \to B.$$

Hence, the maps $gp\colon E \to X$ and $\widetilde{\mathcal{F}}_1\colon E \to E$ together define a map $g'\colon E \to P$ to the pullback in the diagram

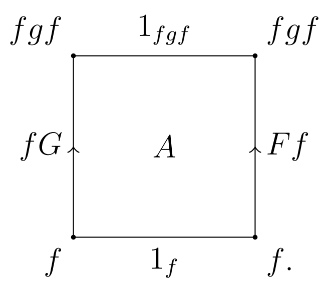


By construction, the map $g'\colon E \to P$ satisfies $f'g' = \widetilde{\mathcal{F}}_1 \simeq \widetilde{\mathcal{F}}_0 = 1_E$.

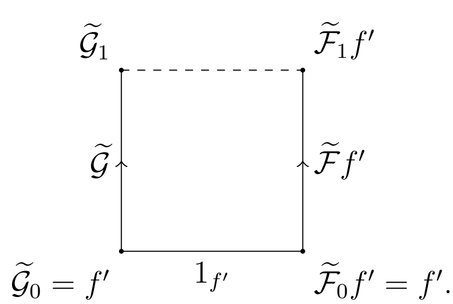
Step 2. By Lemma 2.1, there exists a homotopy $G\colon 1_X \Rightarrow gf$ making $(f, g, G, F^{\text{reverse}})$ into an adjoint homotopy equivalence, i.e., an adjoint equivalence in the track category of spaces, as in Example 2.2. One of the triangle equations, the equality of tracks

$$[fG] = [Ff]\colon f \Rightarrow fgf,$$

says that there is a map $A\colon X \times I^2 \to B$ whose restriction to the boundary $X \times \partial I^2$ is as illustrated in this diagram:



Consider the map $fp' = pf'\colon P \to B$ and the homotopy $fGp'\colon fp' \Rightarrow fgfp'$. Let $\widetilde{\mathcal{G}}\colon P \times I \to E$ be a lift of the homotopy $fGp'\colon P \times I \to B$ beginning with the prescribed lift $\widetilde{\mathcal{G}}_0 = f'\colon P \to E$.
Consider the map $Ap'\colon P \times I^2 \to B$, and let $\widetilde{\mathcal{A}}\colon P \times I^2 \to E$ be a lift of Ap' with prescribed lift as follows. Let $\Lambda \subset I^2$ denote the left, bottom, and right sides of the square I^2, and consider the prescribed lift $P \times \Lambda \to E$ as illustrated in this diagram:



At the top of the square $I \times \{1\} \subset I^2$, the map $p\widetilde{\mathcal{A}} = Ap'$ restricts to the stationary homotopy $1_{fgfp'} = f(1_{gfp'})$. Hence, we obtain a map into the pullback

$$(1_{gfp'}, \widetilde{\mathcal{A}}|_{I\times\{1\}}) \colon P \times I \to X \times_B E = P$$

which provides a homotopy between

$$(gfp', \widetilde{\mathcal{A}}_{0,1}) = (gfp', \widetilde{\mathcal{G}}_1)$$

and

$$(gfp', \widetilde{\mathcal{A}}_{1,1}) = (gpf', \widetilde{\mathcal{F}}_1 f') = (gp, \widetilde{\mathcal{F}}_1)f' = g'f'.$$

On the left side of the square, we obtain a homotopy $(Gp', \widetilde{\mathcal{G}}) \colon P \times I \to P$ between

$$(G_1 p', \widetilde{\mathcal{G}}_1) = (gfp', \widetilde{\mathcal{G}}_1)$$

and

$$(G_0 p', \widetilde{\mathcal{G}}_0) = (p', f') = 1_P,$$

which proves $g'f' \simeq 1_P$. □

Remark 2.4. The statement of Proposition 2.3 can be found in [1, Corollary 1.4]. Also, the proposition says that the Hurewicz model structure on topological spaces is right proper, which follows from the fact that every object is fibrant, i.e., for every space X, the map to a point $X \to *$ is a fibration. More details can be found in [3, Theorem 17.1.1].

Lemma 2.5. *The pullback of a weak homotopy equivalence along a Serre fibration is again a weak homotopy equivalence.*

Proof. Consider the (strict) pullback diagram

$$\begin{array}{ccc} X \times_B E = P & \xrightarrow[\sim ?]{f'} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\sim]{f} & B \end{array}$$

where $p\colon E \to B$ is a Serre fibration and $f\colon X \xrightarrow{\sim} B$ is a weak homotopy equivalence. We want to show that $f'\colon P \to E$ is a weak homotopy equivalence.
The map $p'\colon P \to X$ is a Serre fibration, being the pullback of a Serre fibration. Since the diagram is a strict pullback, the strict fibers (vertically) are isomorphic:

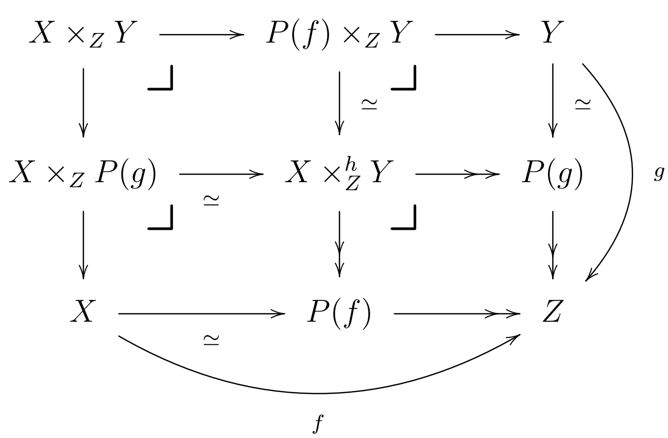
$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ P & \xrightarrow{f'} & E \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\sim]{f} & B. \end{array}$$

By the long exact sequence in homotopy of a Serre fibration, along with the (improved) 5-lemma, f' is a weak homotopy equivalence. □

Proposition 2.6. *1. The natural maps* $P(f) \times_Z Y \xrightarrow{\simeq} X \times^h_Z Y$ *and* $X \times_Z P(g) \xrightarrow{\simeq} X \times^h_Z Y$ *are homotopy equivalences.*

2. If either f *or* g *is a fibration, then the inclusion of the strict pullback into the homotopy pullback* $X \times_Z Y \xrightarrow{\simeq} X \times^h_Z Y$ *is a homotopy equivalence.*

Proof. Consider the following diagram where each square is a strict pullback:



and fibrations are indicated by a double arrowhead. We used the fact that the pullback of a fibration is always a fibration, and the pullback of a homotopy equivalence along a fibration is a homotopy equivalence, by Proposition 2.3. This proves (1).
If $g\colon Y \to Z$ is a fibration, then $X \times_Z Y \xrightarrow{\simeq} P(f) \times_Z Y$ is a homotopy equivalence. If $f\colon X \to Z$ is a fibration, then $X \times_Z Y \xrightarrow{\simeq} X \times_Z P(g)$ is a homotopy equivalence. This proves (2). □

Therefore, it suffices to replace only one of the two maps by a fibration in order to build the homotopy pullback.

Proposition 2.7. *If either* f *or* g *is a Serre fibration, then the inclusion of the strict pullback into the homotopy pullback* $X \times_Z Y \xrightarrow{\simeq} X \times^h_Z Y$ *is a weak homotopy equivalence.*

Proof. Same proof as Proposition 2.6 (2), but using the fact that the pullback of a weak homotopy equivalence along a Serre fibration is a weak homotopy equivalence, Lemma 2.5 . □

3 Homotopy pullback squares

Definition 3.1. A homotopy commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi_Y} & Y \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is called a **homotopy pullback** is there exists a homotopy equivalence $\varphi\colon W \xrightarrow{\simeq} X \times_Z^h Y$ satisfying $\varphi_X = p_X\varphi$ and $\varphi_Y = p_Y\varphi$, as illustrated in the diagram:

$$\begin{array}{cccc} W & & & \\ & \searrow^{\varphi}_{\simeq} & & \overset{\varphi_Y}{\curvearrowright} \\ & X \times_Z^h Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle \varphi_X} & {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ & X & \xrightarrow[f]{} & Z. \end{array}$$

In other words, we allow a more flexible definition of homotopy pullback than the specific construction $X \times_Z^h Y$ introduced in Definition 1.1.

Remark 3.2. A strict pullback diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is also called a **Cartesian** square, because it generalizes the Cartesian product, which is just the pullback over the terminal object:

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow \\ X & \longrightarrow & *. \end{array}$$

In light of this, a homotopy pullback square is also called a **homotopy Cartesian** square.

Example 3.3. By definition, the diagram

$$\begin{array}{ccc} X \times^h_Z Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & \lrcorner\,\text{ho} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a homotopy pullback.

Example 3.4. If either f or g is a fibration, then the strict pullback diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is also a homotopy pullback, by Proposition 2.6.

Example 3.5. For any map $f\colon X \to Y$ and basepoint $y_0 \in Y$, the (homotopy commutative) diagram

$$\begin{array}{ccc} F(f) & \longrightarrow & X \\ \downarrow & \lrcorner\,\text{ho} & \downarrow{\scriptstyle f} \\ * & \xrightarrow[y_0]{} & Y \end{array}$$

is a homotopy pullback.

In particular, the diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & \lrcorner\,\text{ho} & \downarrow{\scriptstyle x_0} \\ * & \xrightarrow[x_0]{} & X \end{array}$$

is a homotopy pullback.

Exercise 3.6 (Pasting lemma for homotopy pullbacks). Consider a homotopy commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

and show the following statements.

1. If both inner squares are homotopy pullbacks, then so is the outer rectangle.
2. If the right-hand square and the outer rectangle are homotopy pullbacks, then so is the left-hand square.

Proposition 3.7. *In a homotopy pullback square of pointed spaces*

$$\begin{array}{ccc} P & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & \lrcorner_{\text{ho}} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

the vertical homotopy fibers are homotopy equivalent: $F(g') \simeq F(g)$*.*
By symmetry, the horizontal homotopy fibers are also homotopy equivalent: $F(f') \simeq F(f)$*.*

Proof. Consider the inclusion of the basepoints of X and Z respectively, which makes the bottom triangle commute:

$$\begin{array}{ccc} P & \xrightarrow{\qquad f' \qquad} & Y \\ {\scriptstyle g'}\downarrow & \lrcorner_{\text{ho}} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\qquad f \qquad]{} & Z \\ & {\scriptstyle x_0}\nwarrow \quad \nearrow{\scriptstyle z_0} & \\ & *. & \end{array}$$

Taking the homotopy pullback of g along the inclusion $* \xrightarrow{z_0} Z$ yields

$$\begin{array}{ccc} F(g) & \longrightarrow & Y \\ \downarrow & \lrcorner_{\text{ho}} & \downarrow{\scriptstyle g} \\ * & \xrightarrow[z_0]{} & Z. \end{array}$$

By commutativity $z_0 = f \circ x_0$, this is equivalent to the same homotopy pullback obtained in two steps:

$$\begin{array}{ccccc} F(g') & \longrightarrow & P & \longrightarrow & Y \\ \downarrow & \lrcorner_{\text{ho}} & {\scriptstyle g'}\downarrow & \lrcorner_{\text{ho}} & \downarrow{\scriptstyle g} \\ * & \xrightarrow[x_0]{} & X & \xrightarrow[f]{} & Z \end{array}$$

which proves the homotopy equivalence $F(g') \simeq F(g)$. $\square$

Remark 3.8. Strict fibers in a strict pullback are isomorphic; homotopy fibers in a homotopy pullback are homotopy equivalent. Mixing the two notions does not work.
Strict fibers in a homotopy pullback are not equivalent in general, for example:

$$\begin{array}{ccc} \Omega X & \not\simeq & * \\ \downarrow & & \downarrow \\ \Omega X & \longrightarrow & * \\ \downarrow & \lrcorner_{\text{ho}} & \downarrow \\ * & \longrightarrow & X. \end{array}$$

Homotopy fibers in a strict pullback are not equivalent in general, for example:

$$\begin{array}{ccc} * & \not\simeq & \Omega X \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & X. \end{array}$$

4 Homotopy invariance

Proposition 4.1. *A homotopy pullback of a homotopy equivalence is a homotopy equivalence.*

Proof. Consider a homotopy pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{f'} & Y \\ {\scriptstyle \simeq ?}\big\downarrow {\scriptstyle g'} & \text{ho} & {\scriptstyle \simeq}\big\downarrow {\scriptstyle g} \\ X & \xrightarrow[f]{} & Z. \end{array}$$

where g is a homotopy equivalence. We want to show that $g' \colon P \to X$ is a homotopy equivalence. By definition of homotopy pullback, and since homotopy equivalences satisfy the 2-out-of-3 property, we may assume P is any of the explicit constructions of $X \times^h_Z Y$, so we assume $P = P(f) \times_Z Y$. In the diagram

$$\begin{array}{ccccc} & & P(f) \times_Z Y & \xrightarrow{f'} & Y \\ & \swarrow^{g'} & \big\downarrow {\scriptstyle p_1} & & {\scriptstyle \simeq}\big\downarrow {\scriptstyle g} \\ X & \xrightarrow[\simeq]{} & P(f) & \overset{}{\underset{\mathrm{ev}_1}{\twoheadrightarrow}} & Z. \end{array} \qquad (f = \mathrm{ev}_1 \circ (X \to P(f)))$$

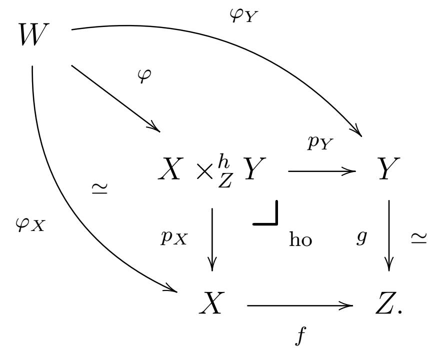
the projection map $p_1 \colon P(f) \times_Z Y \xrightarrow{\simeq} P(f)$ is a homotopy equivalence, being the pullback of the homotopy equivalence g along the fibration ev_1. By 2-out-of-3, g' is a homotopy equivalence. ☐

Corollary 4.2. *Any homotopy commutative square*

$$\begin{array}{ccc} W & \xrightarrow{\varphi_Y} & Y \\ {\scriptstyle \varphi_X}\big\downarrow {\scriptstyle \simeq} & & {\scriptstyle g}\big\downarrow {\scriptstyle \simeq} \\ X & \xrightarrow[f]{} & Z. \end{array}$$

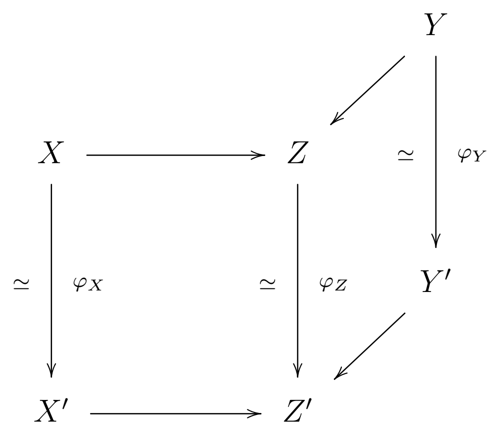
with vertical homotopy equivalences is a homotopy pullback.
By symmetry, the same statement holds for horizontal homotopy equivalences.

Proof. Since the diagram is homotopy commutative, there is a map $\varphi\colon W \to X \times_Z^h Y$ satisfying $p_X\varphi = \varphi_X$ and $p_Y\varphi = \varphi_Y$ in the following diagram:

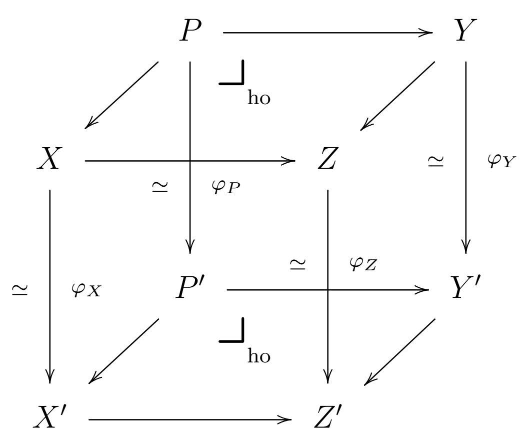


By Proposition 4.1, the projection p_X is a homotopy equivalence. By 2-out-of-3, φ is a homotopy equivalence, so that the original square was a homotopy pullback. ☐

Proposition 4.3 (Homotopy invariance of homotopy pullbacks)**.** *Let φ be a map of diagrams as illustrated here:*



and assume that φ is an objectwise homotopy equivalence. That is, φ_X, φ_Y, and φ_Z are homotopy equivalences. Then the map induced on homotopy pullbacks $\varphi_P\colon X \times_Z^h Y \to X' \times_{Z'}^h Y'$ is a homotopy equivalence, as illustrated here:



Proof. By Corollary 4.2, the right-hand face of the cube is a homotopy pullback. By the "pasting lemma" 3.6, the top face followed by the right-hand face

$$\begin{array}{ccccc} P & \longrightarrow & Y & \xrightarrow{\varphi_Y} & Y' \\ \big\downarrow & \lrcorner_{\text{ho}} & & & \big\downarrow \\ X & \longrightarrow & Z & \xrightarrow{\varphi_Z} & Z' \end{array}$$

form a homotopy pullback. By commutativity of the cube, this "rectangle" is equal to the left-hand face followed by the bottom face

$$\begin{array}{ccccc} P & \longrightarrow & P' & \longrightarrow & Y' \\ \big\downarrow & \lrcorner_{\text{ho}} & & & \big\downarrow \\ X & \xrightarrow{\varphi_X} & X' & \longrightarrow & Z' \end{array}$$

Since the bottom face

$$\begin{array}{ccc} P' & \longrightarrow & Y' \\ \big\downarrow & \lrcorner_{\text{ho}} & \big\downarrow \\ X' & \longrightarrow & Z' \end{array}$$

is a homotopy pullback, then so is the left-hand face

$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle \varphi_P}\big\downarrow & \lrcorner_{\text{ho}} & \big\downarrow{\scriptstyle \varphi_X} \\ P' & \longrightarrow & X' \end{array}$$

by the pasting lemma 3.6. Since φ_X is a homotopy equivalence, so is φ_P by Proposition 4.1. □

5 Weak homotopy invariance

Proposition 5.1. *For any choice of basepoint $z_0 \in Z$, the homotopy pullback $X \times_Z^h Y$ is naturally the total space of a fibration:*

$$\Omega Z \longrightarrow X \times_Z^h Y \xrightarrow{(p_X, p_Y)} X \times Y.$$

Proof. The homotopy pullback can be expressed as

$$X \times_Z^h Y = X \times_Z Z^I \times_Z Y \cong (X \times Y) \times_{(Z \times Z)} Z^I$$

which is the strict pullback of the diagram

$$\begin{array}{ccc} X \times_Z^h Y & \longrightarrow & Z^I \\ {\scriptstyle (p_X, p_Y)} \downarrow & & \downarrow {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ X \times Y & \underset{(f,g)}{\longrightarrow} & Z \times Z. \end{array}$$

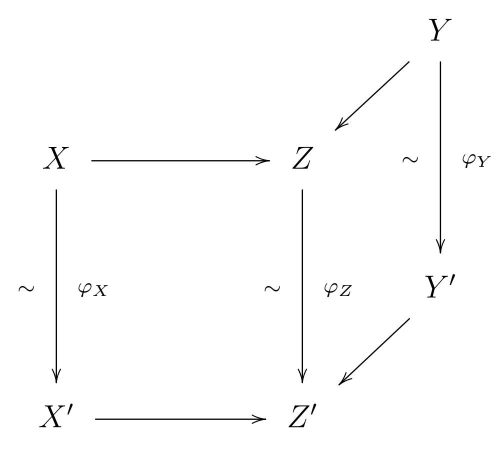
Since $(\mathrm{ev}_0, \mathrm{ev}_1)$ is a fibration, then so is its pullback (p_X, p_Y) and their fibers are equivalent. Said fiber is the loop space $\{\gamma \in Z^I \mid \gamma(0) = \gamma(1) = z_0\} = \Omega Z$. □

Corollary 5.2. *There is a Mayer-Vietoris sequence for the homotopy of a homotopy pullback:*

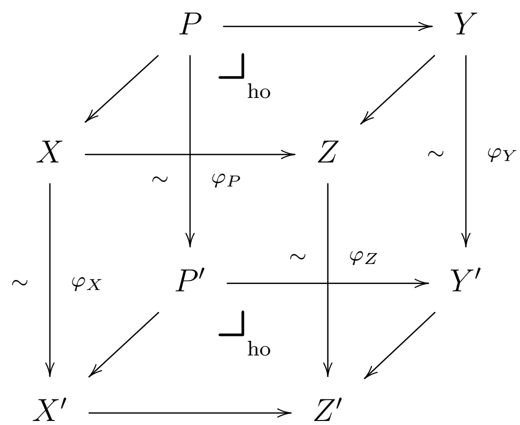
$$\dots \longrightarrow \pi_{n+1}(Z) \longrightarrow \pi_n(X \times_Z^h Y) \xrightarrow{p_{X*} + p_{Y*}} \pi_n(X) \oplus \pi_n(Y) \xrightarrow{f_* - g_*} \pi_n(Z) \longrightarrow \dots$$

Proof. Consider the long exact sequence in homotopy of the fibration from Proposition 5.1. One readily checks that the maps are as claimed. □

Proposition 5.3 (Weak homotopy invariance of homotopy pullbacks)**.** *Let φ be a map of diagrams as illustrated here:*



and assume that φ is an objectwise weak homotopy equivalence. That is, φ_X, φ_Y, and φ_Z are weak homotopy equivalences. Then the map induced on homotopy pullbacks $\varphi_P\colon X \times_Z^h Y \to X' \times_{Z'}^h Y'$ is a weak homotopy equivalence, as illustrated here:



Proof. The induced maps $\varphi_X \times \varphi_Y\colon X \times Y \xrightarrow{\sim} X' \times Y'$ and $\Omega\varphi_Z\colon \Omega Z \xrightarrow{\sim} \Omega Z'$ are weak homotopy equivalences. The induced map of fibrations

$$\begin{array}{ccccc}
\Omega Z & \longrightarrow & X \times_Z^h Y & \xrightarrow{(p_X, p_Y)} & X \times Y \\
{\scriptstyle \sim}\downarrow{\scriptstyle \Omega\varphi_Z} & & \downarrow{\scriptstyle \varphi_P} & & {\scriptstyle \sim}\downarrow{\scriptstyle \varphi_X \times \varphi_Y} \\
\Omega Z' & \longrightarrow & X' \times_{Z'}^h Y' & \xrightarrow{(p_{X'}, p_{Y'})} & X' \times Y'
\end{array}$$

exhibits φ_P as a weak homotopy equivalence. □

References

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