Math 527 - Homotopy Theory
Cofiber sequences

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The notion of cofiber sequence is dual to that of fiber sequence. Most constructions and statements about fiber sequences readily dualize to cofiber sequences, though there are differences to be mindful of.

1. Fiber sequences were defined in the category of pointed spaces $\mathbf{Top}_*$, so that taking the strict fiber (i.e. preimage of the basepoint) makes sense. No such restriction exists for cofiber sequences. There are two different notions: unpointed cofiber sequences in **Top** and pointed cofiber sequences in $\mathbf{Top}_*$.
2. Fiber sequences induce long exact sequences in homotopy, whereas cofiber sequences induce long exact sequences in (co)homology. Therefore the behavior of cofiber sequences with respect to weak homotopy equivalences is a more subtle point.

Warning 0.1. In these notes, we introduce a non-standard notation with $*$ for pointed notions. In the future, as in most of the literature, we will drop the $*$ from the notation and rely on the context to distinguish between pointed and unpointed notions.

1 Definitions

Definition 1.1. The unreduced **cylinder** on a space X is the space $X \times I$.

Note that the inclusion of the base of the cylinder $\iota_0 \colon X \hookrightarrow X \times I$ is a cofibration, as well as a homotopy equivalence.

Definition 1.2. The reduced **cylinder** on a pointed space (X, x_0) is the pointed space $X \wedge I_+ = X \times I / (\{x_0\} \times I)$.

If (X, x_0) is well-pointed, then the inclusion $\iota_0 \colon X \hookrightarrow X \wedge I_+$ is a cofibration; it is always a pointed homotopy equivalence.

If (X, x_0) is well-pointed, then the quotient map $X \times I \twoheadrightarrow X \wedge I_+$ is a homotopy equivalence.

Definition 1.3. Let $f\colon X \to Y$ be a map between spaces. The **unreduced mapping cylinder** of f is the space

$$M(f) = Y \cup_X (X \times I) = Y \amalg (X \times I)/(x,0) \sim f(x)$$

or in other words, the pushout in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota_0} & X \times I \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & M(f) \end{array}$$

in **Top**.

The "inclusion at the top of the cylinder" $\iota_1\colon X \hookrightarrow M(f)$ is always a cofibration.

Definition 1.4. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed map between pointed spaces. The **reduced mapping cylinder** of f is the pointed space

$$M_*(f) = Y \cup_X (X \wedge I_+) = Y \amalg (X \wedge I_+)/(x,0) \sim f(x),$$

in other words the pushout in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota_0} & X \wedge I_+ \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & M_*(f) \end{array}$$

in $\mathbf{Top}_*$ (or equivalently in **Top**).

If (X, x_0) is well-pointed, then the map $\iota_1\colon X \hookrightarrow M_*(f)$ is a cofibration.

The two kinds of mapping cylinder are related as follows:

$$M_*(f) = M(f)/(\{x_0\} \times I)\,.$$

If (X, x_0) is well-pointed, then the quotient map $M(f) \twoheadrightarrow M_*(f)$ is a homotopy equivalence.

Exercise 1.5. Let $\mathcal{C}$ be a complete and cocomplete category, A an object of $\mathcal{C}$, and $(A \downarrow \mathcal{C})$ the category of objects under A, also called "coslice category" or "comma category".

1. Show that $(A \downarrow \mathcal{C})$ is complete and that limits in $(A \downarrow \mathcal{C})$ are the same as in $\mathcal{C}$. More precisely, the forgetful functor $U\colon (A \downarrow \mathcal{C}) \to \mathcal{C}$ preserves limits.

2. Show that $(A \downarrow \mathcal{C})$ is cocomplete and that colimits in $(A \downarrow \mathcal{C})$ are computed as the colimit of the associated diagram in $\mathcal{C}$ with the map $A \to X_j$ to each object X_j.

3. Deduce from (1) that $\mathbf{Top}_*$ is complete, and limits in $\mathbf{Top}_*$ are the same as in $\mathbf{Top}$, which we had already proved using the adjunction

$$(-)_+ : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : U.$$

4. Deduce from (2) that $\mathbf{Top}_*$ is cocomplete, and colimits in $\mathbf{Top}_*$ are computed as the colimit of the associated diagram in $\mathbf{Top}$ with the inclusion of the basepoint $* \hookrightarrow X_j$ of each pointed space X_j in the diagram.

5. Deduce from (4) that coproducts in $\mathbf{Top}_*$ are given by the wedge sum:

$$\coprod_{j \in J} (X_j, x_j) = \bigvee_{j \in J} X_j = \left(\coprod_{j \in J} X_j \right) / \{x_j \mid j \in J\}.$$

6. Deduce from (4) that pushouts in $\mathbf{Top}_*$ are the same as in $\mathbf{Top}$. In other words, the pushout of the diagram

$$\begin{array}{ccc} (W, w_0) & \xrightarrow{g} & (Y, y_0) \\ {\scriptstyle f}\downarrow & & \\ (X, x_0) & & \end{array}$$

in $\mathbf{Top}_*$ is the space $X \amalg Y / f(w) \sim g(w)$ for all $w \in W$, with basepoint $x_0 \sim y_0$.

Definition 1.6. Let X be a space. The unreduced **cone** on X is the space

$$CX = X \times I / (X \times \{1\}).$$

Note that the inclusion of the base of the cone $\iota_0 : X \hookrightarrow CX$ is always a cofibration. In fact, up to reversing the interval I, the map $\iota_0 : X \hookrightarrow CX$ is the map obtained when replacing the map $X \to *$ by a cofibration.

Definition 1.7. Let (X, x_0) be a pointed space. The reduced **cone** on X is the pointed space

$$C_* X = X \wedge I_+ / X \times \{1\} = CX / (\{x_0\} \times I).$$

If (X, x_0) is well-pointed, then the inclusion of the base of the cone $\iota_0 : X \hookrightarrow C_* X$ is a cofibration.

The two kinds of cones are related as follows:

$$C_* X = CX / \{x_0\} \times I.$$

If (X, x_0) is well-pointed, then the quotient map $CX \twoheadrightarrow C_* X$ is a homotopy equivalence.

Definition 1.8. Let $f : X \to Y$ be a map between spaces. The unreduced **mapping cone** (or **cofiber** or **homotopy cofiber**) of f is the space

$$C(f) = M(f) / X \times \{1\} = Y \cup_X CX$$

which is the pushout in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_0}\big\downarrow & \ulcorner & \big\downarrow \\ CX & \longrightarrow & C(f) \end{array}$$

in **Top**. Note in particuliar that the map $i\colon Y \to C(f)$ is automatically a cofibration.

The sequence $X \xrightarrow{f} Y \xrightarrow{i} C(f)$ is called an unpointed **cofiber sequence**.

Remark 1.9. The "homotopy cofiber" is obtained by first replacing $f\colon X \to Y$ by a cofibration using the mapping cylinder $\iota_1\colon X \hookrightarrow M(f)$ and then taking the "strict cofiber", i.e. quotienting out the image

$$C(f) = M(f)/\iota_1(X) = M(f)/X \times \{1\}.$$

Remark 1.10. Technically, in the category of Hausdorff spaces or CGWH spaces, the "strict cofiber" is given by quotienting out the *closure* of the image. This does not affect remark 1.9: in the category of CGWH spaces, every cofibration is closed.

Example 1.11. Let $f\colon S^{n-1} \to X$ be a map. The space obtained from X by attaching an n-cell using f as attaching map is the cofiber of f:

$$S^{n-1} \xrightarrow{f} X \hookrightarrow X \cup_f e^n = C(f).$$

Attaching an arbitrary set of cells is obtained the same way. If $\{f_j\colon S^{n_j-1} \to X\}_{j\in J}$ is a collection of maps, then the space obtained from X by attaching cells with those attaching maps f_j is the cofiber

$$\coprod_{j\in J} S^{n_j-1} \xrightarrow{f=(f_j)_{j\in J}} X \hookrightarrow X \cup_f \bigcup_{j\in J} e^{n_j} = C(f).$$

Definition 1.12. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between pointed spaces. The reduced **mapping cone** (or **cofiber** or **homotopy cofiber**) of f is the space

$$C_*(f) = M_*(f)/X \times \{1\} = Y \cup_X C_*X$$

which is the pushout in the diagram

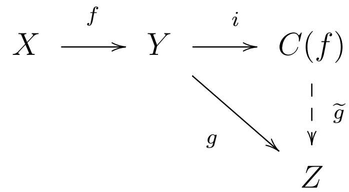
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_0}\big\downarrow & \ulcorner & \big\downarrow \\ C_*X & \longrightarrow & C_*(f) \end{array}$$

in $\textbf{Top}_*$ (or equivalently, in **Top**). If (X, x_0) is well-pointed, then the map $\iota_0\colon X \to C_*X$ is a cofibration, and so is $i\colon Y \to C_*(f)$.

The sequence $X \xrightarrow{f} Y \xrightarrow{i} C_*(f)$ is called a pointed **cofiber sequence**.

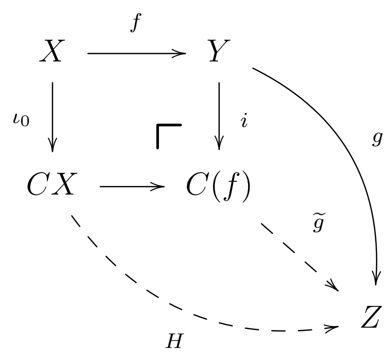
2 Basic properties

Proposition 2.1. *Let Z be a space and $g\colon Y \to Z$ a map between spaces. Consider the extension problem*



in **Top**. *Then extensions $\widetilde{g}\colon C(f) \to Z$ of g correspond bijectively to null-homotopies of the composite $gf\colon X \to Z$. In particular, an extension exists if and only if the restriction gf is null-homotopic.*

Proof. Consider the pushout diagram defining $C(f)$. Extensions of g

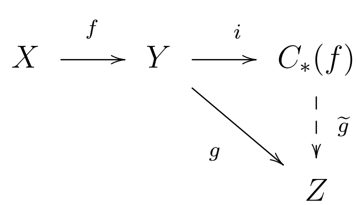


correspond to maps $H\colon CX \to Z$ making the diagram above commute, i.e. satisfying $\iota_0 \circ H = gf$. These are precisely null-homotopies of gf. □

Remark 2.2. If (Z, z_0) is pointed and we use the "vertex of the cone" as basepoint for $C(f)$, then *pointed* extensions $\widetilde{g}\colon C(f) \to Z$ of g correspond bijectively to homotopies of the composite $gf\colon X \to Z$ to the null map (i.e. constant with value z_0).

The same argument proves the pointed analogue.

Proposition 2.3. *Let Z be a pointed space and $g\colon Y \to Z$ a pointed map between pointed spaces. Consider the extension problem*



in $\mathbf{Top}_*$. *Then extensions $\widetilde{g}\colon C_*(f) \to Z$ of g correspond bijectively to pointed null-homotopies of the composite $gf\colon X \to Z$. In particular, an extension exists if and only if the restriction gf is pointed null-homotopic.*

The particular case of the statement can be reinterpreted as follows.

Corollary 2.4. *For any pointed space Z, applying the functor $[-,Z]_*\colon \mathbf{Top}_* \to \mathbf{Set}_*$ to the pointed cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} C_*(f)$ yields*

$$[C_*(f),Z]_* \xrightarrow{i^*} [Y,Z]_* \xrightarrow{f^*} [X,Z]_*$$

which is an exact sequence of pointed sets.

In fact, this statement is the π_0 shadow of a stronger statement at the level of mapping spaces.

Proposition 2.5. *Let $f\colon X \to Y$ be a map between spaces, and Z any space. Then applying the functor* $\mathrm{Map}(-,Z)\colon \mathbf{Top} \to \mathbf{Top}$ *to the mapping cylinder $M(f)$ yields the path space construction on the induced (restriction) map $f^*\colon Z^Y \to Z^X$.*

Moreover, the homeomorphism $Z^{M(f)} \cong P(f^*)$ is compatible with the usual cofibration - homotopy equivalence factorization

$$X \xrightarrow{\iota_1} M(f) \xrightarrow{p} Y$$

so that applying $\mathrm{Map}(-,Z)$ *to the latter yields the usual homotopy equivalence - fibration factorization*

$$Z^Y \xrightarrow{i} P(f^*) \xrightarrow{\mathrm{ev}_1} Z^X.$$

Proof. Recall that the mapping cylinder $M(f)$ is the pushout $Y \cup_f X \times I$, and that $\mathrm{Map}(-,Z)$ sends coproducts to products in an enriched sense. Thus we obtain the homeomorphism

$$\begin{aligned}
\mathrm{Map}\,(M(f),Z) &= \mathrm{Map}\,(Y \cup_f X \times I, Z) \\
&\cong \mathrm{Map}\,(Y,Z) \times_{f^*} \mathrm{Map}\,(X \times I, Z) \\
&\cong \mathrm{Map}\,(Y,Z) \times_{f^*} \mathrm{Map}\left(I, Z^X\right) \\
&= Z^Y \times_{f^*} (Z^X)^I \\
&= P(f^*).
\end{aligned}$$

Via this homeomorphism, the dual of "inclusion at 1" $\iota_1\colon X \to M(f)$ is "evaluation at 1" $\mathrm{ev}_1\colon Z^{M(f)} \cong P(f^*) \to Z^X$. The dual of "collapsing the mapping cylinder" $p\colon M(f) \to Y$ is "embedding constant paths" $i\colon Z^Y \to Z^{M(f)} \cong P(f^*)$. □

Remark 2.6. The homeomorphism $Z^{M(f)} \cong P(f^*)$ was the key ingredient in showing that applying $\mathrm{Map}(-,Z)$ to a cofibration $i\colon A \to X$ yields a fibration $i^*\colon Z^X \to Z^A$.

There is a pointed analogue of proposition 2.5.

Proposition 2.7. *Let $f\colon X \to Y$ be a pointed map between pointed spaces, and Z any pointed space.*

1. *Applying the functor* $\mathrm{Map}_*(-,Z)\colon \mathbf{Top}_* \to \mathbf{Top}_*$ *to the reduced mapping cylinder* $M_*(f)$ *yields the path space construction on the induced (restriction) map* $f^*\colon Z^Y \to Z^X$.

   Moreover, the homeomorphism $Z^{M_*(f)} \cong P(f^*)$ *is compatible with the usual factorization* $X \xrightarrow{\iota_1} M_*(f) \xrightarrow{p} Y$, *so that applying* $\mathrm{Map}_*(-,Z)$ *to the latter yields the usual factorization*

$$Z^Y \xrightarrow{i} P(f^*) \xrightarrow{\mathrm{ev}_1} Z^X.$$

2. *Applying* $\mathrm{Map}_*(-,Z)$ *to the reduced mapping cone* $C_*(f)$ *yields (up to a sign in the interval* I*) the homotopy fiber of the induced map* $F(f^*)$*, in a way that is compatible with the fiber sequence. More precisely, applying* $\mathrm{Map}_*(-,Z)$ *to the pointed cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} C_*(f)$$

   yields the fiber sequence

$$\begin{array}{ccccc} Z^{C_*(f)} & \xrightarrow{i^*} & Z^Y & \xrightarrow{f^*} & Z^X \\ \| & \nearrow_{p} & & & \\ F(f^*). & & & & \end{array}$$

Proof. (1) Similar to the unpointed case, but using the natural homeomorphisms

$$\begin{aligned} \mathrm{Map}_*\left(X \wedge I_+, Z\right) &\cong \mathrm{Map}_*\left(I_+, Z^X\right) \\ &\cong \mathrm{Map}\left(I, Z^X\right). \end{aligned}$$

(2) Recall that the reduced mapping cone $C_*(f)$ is the pushout $Y \cup_f C_*X$ in $\mathbf{Top}_*$, and the reduced cone on X can be viewed as the smash product

$$C_*X = X \wedge I$$

where I is given the basepoint $1 \in I$. Thus we obtain the homeomorphism

$$\begin{aligned} \mathrm{Map}_*\left(C_*(f), Z\right) &= \mathrm{Map}_*\left(Y \cup_f C_*X, Z\right) \\ &\cong \mathrm{Map}_*\left(Y, Z\right) \times_{f^*} \mathrm{Map}_*\left(C_*X, Z\right) \\ &= \mathrm{Map}_*\left(Y, Z\right) \times_{f^*} \mathrm{Map}_*\left(X \wedge I, Z\right) \\ &\cong \mathrm{Map}_*\left(Y, Z\right) \times_{f^*} \mathrm{Map}_*\left((I,1), Z^X\right) \\ &\cong \mathrm{Map}_*\left(Y, Z\right) \times_{f^*} \mathrm{Map}_*\left((I,0), Z^X\right) \\ &= Z^Y \times_{f^*} P(Z^X) \text{ where paths in } Z^X \text{ start at the basepoint} \\ &= F(f^*). \end{aligned}$$

Note: We flipped the interval I because of our convention that the path space consists of paths that *start* at the basepoint.

Via this homeomorphism, the dual of the inclusion $i\colon Y \to C_*(f)$ is the natural projection $p\colon Z^{C_*(f)} \cong F(f^*) \to Z^Y$ onto the "first factor" in the expression $F(f^*) \cong Z^Y \times_{f^*} P(Z^X)$. □

Thus, the exact sequence of pointed sets 2.4 can be viewed as the exact sequence obtained by applying π_0 (i.e. taking path components) to the fiber sequence

$$Z^{C_*(f)} \xrightarrow{i^*} Z^Y \xrightarrow{f^*} Z^X.$$

There is a canonical "quotient of the homotopy cofiber onto the strict cofiber" $q\colon C(f) \twoheadrightarrow Y/\overline{f(X)}$ defined by quotienting out the closure $\overline{CX} \subseteq C(f) = Y \cup_f CX$.

Proposition 2.8. *If $f\colon X \to Y$ is a cofibration, then the canonical map $q\colon C(f) \twoheadrightarrow Y/\overline{f(X)}$ is a homotopy equivalence.*

Proof. The natural map $CX \to C(f) = Y \cup_f CX$ is a cofibration, being the pushout along $X \to CX$ of the cofibration $f\colon X \to Y$. Moreover, the cone CX is contractible. Therefore the quotient map $q\colon C(f) \twoheadrightarrow C(f)/CX$ is a homotopy equivalence (c.f. Hatcher Proposition 0.17).

Since we are working in CGWH spaces, $f(X) \subseteq Y$ is necessarily closed in Y. To conclude, note the homeomorphism $C(f)/CX \cong Y/f(X)$. □

Remark 2.9. The reduced mapping cone $C_*(f)$ is rarely a cokernel of f in the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$.

Exercise 2.10. Consider the "multiplication by 2" map $f\colon S^1 \to S^1$. For instance, viewing $S^1 \subset \mathbb{C}$ as the unit circle in the complex plane, the map f can be realized as $f(z) = z^2$.

Show that the map f does *not* admit a cokernel in $\mathrm{Ho}(\mathbf{Top}_*)$.

Remark 2.11. This shows in particular that $\mathrm{Ho}(\mathbf{Top}_*)$ is not cocomplete, though it does have all small coproducts, which are given by the wedge sum as in $\mathbf{Top}_*$.

3 Homotopy invariance

Note that taking the (unreduced) cofiber is functorial in the input $f\colon X \to Y$, i.e. is a functor $\mathrm{Arr}(\mathbf{Top}) \to \mathbf{Top}$ from the arrow category of **Top**, and such that the map $Y \to C(f)$ is a natural transformation. Thus, a map of diagrams

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right),$$

which is a (strictly) commutative diagram in **Top**

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ X' & \xrightarrow[f']{} & Y' \end{array} \tag{1}$$

induces a map between cofibers $\varphi_C\colon C(f) \to C(f')$ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C(f) \\ {\scriptstyle \varphi_X}\downarrow & & {\scriptstyle \varphi_Y}\downarrow & & \downarrow{\scriptstyle \varphi_C} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[i']{} & C(f') \end{array} \tag{2}$$

in **Top** commute. Moreover, this assignment preserves compositions (as in “stacking another square” below the left-hand square).

The analogous statements holds for the reduced cofiber $C_*(f)$.

Let us study to what extent the cofiber is a homotopy invariant construction.

Proposition 3.1. *Consider a map of diagrams*

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right)$$

in **Top**.

1. *If φ is an objectwise homotopy equivalence, i.e. both maps $\varphi_X\colon X \xrightarrow{\simeq} X'$ and $\varphi_Y\colon Y \xrightarrow{\simeq} Y'$ are homotopy equivalences, then the induced map on unreduced cofibers $\varphi_C\colon C(f) \to C(f')$ is also a homotopy equivalence.*

2. *If φ is an objectwise weak homotopy equivalence, i.e. both maps $\varphi_X\colon X \xrightarrow{\sim} X'$ and $\varphi_Y\colon Y \xrightarrow{\sim} Y'$ are weak homotopy equivalences, then the induced map on cofibers $\varphi_C\colon C(f) \to C(f')$ is also a weak homotopy equivalence.*

Proof. (1) Dualize the proof of the analogous statement for homotopy fibers.

(2) Trickier. We will come back to it. □

Remark 3.2. Proposition 3.1 holds more generally for homotopy pushouts.

The pointed analogues also hold, for similar reasons.

Proposition 3.3. *Consider a map of diagrams*

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right)$$

in $\mathbf{Top}_*$.

1. *If* φ *is an objectwise pointed homotopy equivalence, i.e. both maps* $\varphi_X\colon X \xrightarrow{\simeq} X'$ *and* $\varphi_Y\colon Y \xrightarrow{\simeq} Y'$ *are pointed homotopy equivalences, then the induced map on reduced cofibers* $\varphi_C\colon C_*(f) \to C_*(f')$ *is also a pointed homotopy equivalence.*

2. *If* φ *is an objectwise weak homotopy equivalence, i.e. both maps* $\varphi_X\colon X \xrightarrow{\sim} X'$ *and* $\varphi_Y\colon Y \xrightarrow{\sim} Y'$ *are weak homotopy equivalences, then the induced map on reduced cofibers* $\varphi_C\colon C_*(f) \to C_*(f')$ *is also a weak homotopy equivalence.*

4 Iterated cofiber sequence

Proposition 4.1. *Consider the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} C(f)$. Then the quotient map $\varphi \colon C(i) \twoheadrightarrow SX$ of the homotopy cofiber of i onto its strict cofiber is a homotopy equivalence making the following diagram commute:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C(f) & \xrightarrow{q} & SX \\ & & & & & \searrow^{i'} & \uparrow \varphi \ \simeq \\ & & & & & & C(i) \end{array}$$

where $q \colon C(f) \twoheadrightarrow SX \cong C(f)/i(Y)$ is the quotient map.

Proof. The strict cofiber of i is the quotient

$$C(f)/i(Y) = (Y \cup_f CX)/Y = CX/(X \times \{0\}) \cong SX$$

which is the unreduced suspension of X. By construction, the composite $\varphi \circ i' \colon C(f) \to SX$ is the quotient map q.

The quotient map $\varphi \colon C(i) \twoheadrightarrow SX$ is a homotopy equivalence, since i is a cofibration (and using 2.8). □

In light of the proposition, the sequence $Y \to C(f) \to SX$ is sometimes also called a cofiber sequence.

Proposition 4.2. *Consider the pointed cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} C_*(f)$ and assume X is well-pointed. Then the quotient map $\varphi \colon C_*(i) \twoheadrightarrow \Sigma X$ of the homotopy cofiber of i onto its strict cofiber is a pointed homotopy equivalence making the following diagram commute:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C_*(f) & \xrightarrow{q} & \Sigma X \\ & & & & & \searrow^{i'} & \uparrow \varphi \ \simeq \\ & & & & & & C_*(i) \end{array}$$

where $q \colon C_*(f) \twoheadrightarrow \Sigma X \cong C_*(f)/i(Y)$ is the quotient map, and ΣX denotes the reduced suspension of X.

For the purposes of iterating the cofiber construction, we will focus on the pointed case. Also assume that X and Y are well-pointed.

Proposition 4.3. *The following triangle*

$$\begin{array}{ccc} \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\ \simeq \uparrow \varphi & \nearrow_{q'} & \\ C_*(i) & & \end{array}$$

commutes up to homotopy. Here, the map $q' \colon C_*(i) \to \Sigma Y$ is the quotient map

$$C_*(i) = C_*(f) \cup_i C_*Y \twoheadrightarrow (C_*(f) \cup_i C_*Y)/C_*(f) \cong C_*Y/(Y \times \{0\}) \cong \Sigma Y.$$

See May § 8.4 for more details.

Definition 4.4. The (long) **cofiber sequence** generated by a pointed map $f \colon X \to Y$ between well-pointed spaces is the sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_*(f) \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_*(f) \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \to \dots \tag{3}$$

where $i \colon Y \to C_*(f)$ and $q \colon C_*(f) \to \Sigma X$ are defined above.

Such a sequence is sometimes called a **Puppe sequence**.

Proposition 4.5. *Let $f \colon X \to Y$ be a pointed map, and Z any pointed space. Then applying the functor $[-, Z]_* \colon$ **Top**$_* \to$ **Set**$_*$ to the cofiber sequence generated by f yields*

$$\dots \to [\Sigma^2 X, Z]_* \to [\Sigma C_*(f), Z]_* \to [\Sigma Y, Z]_* \to [\Sigma X, Z]_* \to [C_*(f), Z]_* \to [Y, Z]_* \to [X, Z]_*$$

which is a long exact sequence of pointed sets.

Proof. By 4.2 and 4.3, each consecutive three spots of the long cofiber sequence form, up to homotopy equivalence, a pointed cofiber sequence. The result follows from 2.4. □

Note that $[\Sigma X, Z]_*$ is naturally a group and $[\Sigma^2 X, Z]_*$ is naturally an abelian group.

The following proposition is not a consequence of 4.5, but realizes a familiar long exact sequence topologically.

Proposition 4.6. *Let $i \colon A \to X$ be a pointed map between well-pointed spaces. Then applying reduced homology $\widetilde{H}_n(-)$ to the cofiber sequence generated by i yields, up to signs, the long exact sequence in homology*

$$\widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X, A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \to \dots \tag{4}$$

In particular, taking n large enough, one recovers the entire long exact sequence of the pair (X, A).

Proof. For a well-pointed space X, one has the suspension isomorphism $\widetilde{H}_k(X) \cong \widetilde{H}_{k+1}(\Sigma X)$ for all $k \geq -1$. Moreover, since i is a pointed map between well-pointed spaces, the relative homology groups are $\widetilde{H}_k(X, A) \cong \widetilde{H}_k(C_*(i))$. The long exact sequence of the "good pair" $(M_*(i), A)$ (c.f. Hatcher Theorem 2.13) coincides with the usual long exact sequence (4) of the pair (X, A), up to signs. □

Proposition 4.7. *Let $i\colon A \to X$ be a pointed map between well-pointed spaces. Then there is a long exact sequence in reduced cohomology*

$$\ldots \to \widetilde{H}^{n-1}(A; M) \xrightarrow{d} \widetilde{H}^n(X, A; M) \xrightarrow{j^*} \widetilde{H}^n(X; M) \xrightarrow{i^*} \widetilde{H}^n(A) \xrightarrow{d} \widetilde{H}^{n+1}(X, A; M) \to \ldots$$

with coefficients in any abelian group M.

Proof. Reduced cohomology $\widetilde{H}^n(-; M)$ is represented (at least for spaces with the homotopy type of a CW-complex) by the Eilenberg-MacLane space $K(M, n)$, that is:

$$\widetilde{H}^n(X; M) \cong [X, K(M, n)]_*.$$

Applying the functor $[-, K(M, n)]_*$ to the cofiber sequence generated by $i\colon A \to X$ yields a long exact sequence

$$\ldots \to [\Sigma X, K(M, n)]_* \to [\Sigma A, K(M, n)]_* \to [C_*(i), K(M, n)]_* \to [X, K(M, n)]_* \to [A, K(M, n)]_*. \tag{5}$$

Now use the natural isomorphism

$$[\Sigma X, K(M, n)]_* \cong [X, \Omega K(M, n)]_*$$

and the equivalence

$$\Omega K(M, n) \simeq K(M, n-1)$$

for $n \geq 1$ and $\Omega K(M, 0) \simeq *$.

For a well-pointed space X, one has the suspension isomorphism $\widetilde{H}^k(X; M) \cong \widetilde{H}^{k+1}(\Sigma X; M)$ for all $k \geq -1$. Moreover, since $i\colon A \to X$ is a pointed map between well-pointed spaces, the relative cohomology groups are $\widetilde{H}^k(X, A; M) \cong \widetilde{H}^k(C_*(i); M)$. Taking n large enough, the long exact sequence (5) yields the long exact sequence in the statement. □

Note that the long exact sequence (5) coincides with the usual long exact sequence of a pair, up to certain signs of the maps.