Math 527 - Homotopy Theory
Exactness in low degrees

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Recall that a Serre fibration $F \xrightarrow{\iota} E \xrightarrow{p} B$ induces a long exact sequence of homotopy groups:

$$\ldots \pi_2(B) \xrightarrow{\partial} \pi_1(F) \xrightarrow{\iota_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{\iota_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).$$

Although the last three terms are not groups, there is an action of $\pi_1(B)$ on $\pi_0(F)$ such that exactness behaves as follows.

Exactness at $\pi_1(B)$ takes the form:

$$\partial(\alpha) = \partial(\beta) \;\Leftrightarrow\; \alpha\beta^{-1} \in \mathrm{im}(p_*)$$

or written more suggestively as $\alpha = p_*(\gamma)\beta$ for some $\gamma \in \pi_1(E)$. This follows from the formula $\partial(\alpha) = \alpha \cdot e_0$, where $e_0 \in F$ is the basepoint of E (and hence of F).

Exactness at $\pi_0(F)$ takes the form:

$$\iota_*(f) = \iota_*(f') \;\Leftrightarrow\; f = \alpha \cdot f'$$

for some $\alpha \in \pi_1(B)$.

Remark 0.1. Given a path α in B from b_0 to b_1, consider the induced change-of-fiber map in the forward direction $\alpha_* \colon F_{b_0} \to F_{b_1}$. Multiplication in $\pi_1(B)$, or more generally concatenation of paths in $\Pi_1(B)$, is such that the path $\alpha\beta$ goes through α first and then through β. Because we write composition of maps by "applying maps from right to left", this results in the formula

$$(\alpha\beta)_* = \beta_* \circ \alpha_*$$

(c.f. Hatcher Proposition 4.61, where he denotes our α_* by L_α), and thus a *right* action of $\pi_1(B)$ on $\pi_0(F)$.

Instead, we use the following sign convention. Consider the induced change-of-fiber map in the reverse direction $\alpha^* := (\alpha^{-1})_* \colon F_{b_1} \to F_{b_0}$, which now satisfies the formula

$$(\alpha\beta)^* = \alpha^* \circ \beta^*.$$

Define the action by $\alpha \cdot f := \alpha^*(f)$, which results in a *left* action of $\pi_1(B)$ on $\pi_0(F)$:

$$\begin{aligned}\alpha \cdot (\beta \cdot f) &= (\alpha^* \circ \beta^*)(f) \\ &= (\alpha\beta)^*(f) \\ &= (\alpha\beta) \cdot f.\end{aligned}$$

Hatcher uses the left action convention for the action of $\pi_1(X)$ on $\pi_n(X)$ (c.f. § 4.1.Definitions and Basic Constructions), but the right action convention for the action of $\pi_1(B)$ on $\pi_0(F)$ (c.f. beginning of § 4.A).