Math 527 - Homotopy Theory
Additional notes

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February 11, 2013

1 Well-pointed spaces

Definition 1.1. A pointed space (X, x_0) is **well-pointed** or **non-degenerately based** if the inclusion of the basepoint $\{x_0\} \hookrightarrow X$ is a cofibration.

Example 1.2. Any CW-complex based at a 0-cell is well-pointed.

Non-example 1.3. The space $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ based at 0 is *not* well-pointed.

Proposition 1.4. *Let $f \colon (X, x_0) \to (Y, y_0)$ be a map (not necessarily pointed) between well-pointed spaces. If $f(x_0)$ is in the path component of y_0, then f is homotopic to a pointed map. Moreover, the homotopy $H \colon X \times I \to Y$ can be chosen so that its restriction to $\{x_0\} \times I$ is any prescribed path in Y from $f(x_0)$ to y_0.*

Proof. Let $\gamma \colon I \to Y$ be a path from $f(x_0)$ to y_0. Consider the map $f \colon X \to Y$, and the homotopy of its restriction to the basepoint $\{x_0\} \times I \cong I \xrightarrow{\gamma} Y$. Since the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration, this homotopy can be extended to $\widetilde{H} \colon X \times I \to Y$. Thus $f = \widetilde{H}_0$ is homotopic to $\widetilde{H}_1$, which is a pointed map:

$$\widetilde{H}_1(x_0) = \widetilde{H}(x_0, 1) = \gamma(1) = y_0.$$

$\square$

Lemma 1.5. *Let $i \colon (A, a_0) \to (X, x_0)$ be a pointed map between well-pointed spaces. If i is a based cofibration, then the map $i \times \mathrm{id} \colon A \times I \to X \times I$ is a based cofibration.*

Here, the basepoint of I is, say, 0, so that $A \times I$ has basepoint $(a_0, 0)$ and $X \times I$ has basepoint $(x_0, 0)$.

Proposition 1.6. *Let $i \colon (A, a_0) \to (X, x_0)$ be a pointed map between well-pointed spaces. If i is a based cofibration, then i is also an unbased cofibration.*

Proof. Consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\;H\;} & Y^I \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

where $f\colon X \to Y$ is an arbitrary map. The space Y does not come with a specified basepoint, so we choose $y_0 := f(x_0)$ as basepoint. Then $f\colon X \to Y$ is a pointed map, though $H\colon A \to Y^I$ is (usually) not.

Changing H to a pointed map. Note that $H(a_0) \in Y^I$ is a path starting at y_0:

$$H(a_0)(0) = (\mathrm{ev}_0 \circ H)(a_0) = (f \circ i)(a_0) = f(x_0) = y_0.$$

Consider a path $\Gamma\colon I \to Y^I$ from $H(a_0)$ to c_{y_0}, the constant path at y_0 (which is the basepoint of Y^I), where $\Gamma(s)$ always starts at y_0. For example, one could shrink $H(a_0)$ using the formula $\Gamma(s)(t) = H(a_0)((1-s)t)$.

Applying proposition 1.4 to the map $H\colon A \to Y^I$ and the path Γ, one obtains a homotopy $\widetilde{H}\colon A \times I \to Y^I$ satisfying:

$$\widetilde{H}(a,0) = H(a) \text{ for all } a \in A$$

$$\widetilde{H}(a_0,-) = \Gamma$$

$$\widetilde{H}(-,1)\colon A \to Y^I \text{ is pointed.}$$

Writing $G(a,s,t) := \widetilde{H}(a,s)(t)$, we obtain a map $G\colon A \times I \times I \to Y$ satisfying the three conditions:

$$G(a,0,t) = H(a)(t) \tag{1}$$

$$G(a_0,s,0) = y_0 \text{ for all } s \in I \tag{2}$$

$$G(a_0,1,t) = y_0 \text{ for all } t \in I. \tag{3}$$

Condition (2) says that the map $F\colon A \times I \to Y$ defined by $F(a,s) = G(a,s,0)$, i.e. $F = \mathrm{ev}_0 \circ \widetilde{H}$, is a pointed homotopy. Moreover, it satisfies $F(a,0) = f(i(a))$. Since i is a based cofibration, the pointed homotopy F can be extended to a (pointed) homotopy $\widetilde{F}\colon X \times I \to Y$ satisfying $\widetilde{F}(x,0) = f(x)$ for all $x \in X$. In other words, there is a lift $\widetilde{F}$ in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;F\;} & Y^I \\ {\scriptstyle i}\big\downarrow & \overset{\widetilde{F}}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow[\;f\;]{} & Y. \end{array}$$

Recovering H. Viewing $\widetilde{F}$ as a map $\widetilde{F}\colon X \times I \to Y$ and taking the basepoint $(x_0, 1) \in X \times I$, note that this map is pointed:

$$\widetilde{F}(x_0, 1) = \widetilde{F}(i(a_0), 1) = F(a_0, 1) = y_0.$$

Now think of $s \in I$ as a "space parameter" and $t \in I$ as a "time parameter", so that the map

$$G\colon (A \times I) \times I \to Y$$

is a homotopy starting at $G(-, -, 0) = F$. This homotopy G is pointed, by condition (3), using $(a_0, 1) \in A \times I$ as basepoint. By lemma 1.5, the pointed homotopy G can be extended to a (pointed) homotopy $\widetilde{G}\colon X \times I \times I \to Y$ satisfying $\widetilde{G}(x, s, 0) = \widetilde{F}(x, s)$.

At $s = 0$, $\widetilde{G}(-, 0, -)$ provides the desired extension of H:

$$\widetilde{G}(i(a), 0, t) = G(a, 0, t) = H(a)(t)$$

while agreeing with the original map $f\colon X \to Y$ at $t = 0$:

$$\widetilde{G}(x, 0, 0) = \widetilde{F}(x, 0) = f(x).$$

□

Proposition 1.7. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed map between well-pointed spaces. If f is a homotopy equivalence, then f is also a pointed homotopy equivalence.*