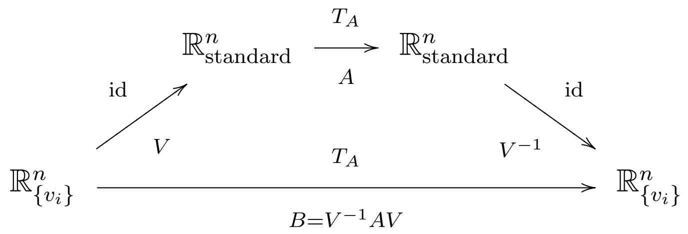
Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Similar matrices

1 Change of basis

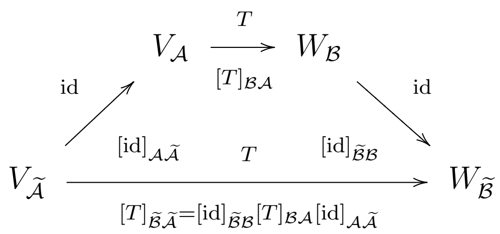
Consider an $n \times n$ matrix A and think of it as the standard representation of a transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$. If we pick a different basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$, what matrix B represents T_A with respect to that new basis?

Write $V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ and consider the diagram



which says the new matrix is $B = V^{-1}AV$.

Remark: This is an instance of the more general change of coordinates formula. Start with a linear transformation $T \colon V \to W$. Let $\mathcal{A}$ be the "old" basis of V and $\widetilde{A}$ the "new" basis. Let $\mathcal{B}$ be the "old" basis of W and $\widetilde{B}$ the "new" basis. Then the diagram



gives the change of coordinates formula

$$[T]_{\widetilde{\mathcal{B}}\widetilde{\mathcal{A}}} = [\mathrm{id}]_{\widetilde{\mathcal{B}}\mathcal{B}}[T]_{\mathcal{B}\mathcal{A}}[\mathrm{id}]_{\mathcal{A}\widetilde{\mathcal{A}}}.$$

Example: Let $A = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$, viewed as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Find the matrix B representing the same transformation with respect to the basis $\{v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

Write $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$. Then we have

$$\begin{aligned} B &= V^{-1}AV \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 12 & -1 \\ 4 & -2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

What this means is that the transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is determined by

$$\begin{aligned} Av_1 &= 4v_1 + 0v_2 = 4v_1 \ \ \text{(first column of } B\text{)} \\ Av_2 &= 0v_1 - 1v_2 = -v_2 \ \ \text{(second column of } B\text{).} \end{aligned}$$

Let us check this explicitly:

$$\begin{aligned} Av_1 &= \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 4v_1 \\ Av_2 &= \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -v_2. \end{aligned}$$

2 Similar matrices

The change of basis formula $B = V^{-1}AV$ suggests the following definition.

Definition: A matrix B is **similar** to a matrix A if there is an invertible matrix S such that $B = S^{-1}AS$.

In particular, A and B must be square and A, B, S all have the same dimensions $n \times n$. The idea is that matrices are similar if they represent the same transformation $V \to V$ up to a change of basis.

Example: In the example above, we have shown that $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ is similar to $\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$.

Exercise: Show that similarity of matrices is an equivalence relation.

3 Trace of a matrix

How can we tell if matrices are similar? There is an easy necessary (though not sufficient) condition.

Definition: The **trace** of an $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = a_{1,1} + a_{2,2} + \ldots + a_{n,n}.$$

Examples: $\operatorname{tr} \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix} = 6, \qquad \operatorname{tr} \begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix} = 9, \qquad \operatorname{tr} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 15,$

$$\operatorname{tr} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3, \qquad \operatorname{tr} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4.$$

Proposition: For any two $n \times n$ matrices A and B, we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof:

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} \\ &= \sum_{k=1}^{n} (BA)_{kk} \\ &= \operatorname{tr}(BA). \blacksquare \end{aligned}$$

Example:

$$\operatorname{tr}\left(\begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix} \right) = \operatorname{tr} \begin{bmatrix} 55 & 3 \\ -4 & -4 \end{bmatrix} = 51$$

$$\operatorname{tr}\left(\begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix} \right) = \operatorname{tr} \begin{bmatrix} 38 & 54 \\ 13 & 13 \end{bmatrix} = 51.$$

Corollary: Similar matrices have the same trace.

Proof: Homework #8.5.

Example: In the example above, we have $\operatorname{tr}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} = 3$ and $\operatorname{tr}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = 3$.

Warning! The converse does **not** hold. In other words, matrices with the same trace are rarely similar.

Example: The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

all have trace 0, but none of them is similar to another, since they have different ranks (respectively 0, 1, 2).

In fact, for any number $a \in \mathbb{R}$ the matrix $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ has trace 0. However, if we pick a number $b \neq \pm a$, then $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ is **not** similar to $\begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}$.

In chapter 4, we will learn better tools to tell whether or not two matrices are similar.