1 Planar rotations

Definition: A planar rotation in $\mathbb{R}^n$ is a linear map $R: \mathbb{R}^n \to \mathbb{R}^n$ such that there is a plane $P \subseteq \mathbb{R}^n$ (through the origin) satisfying

$$R(P) \subseteq P \text{ and } R|_P = \text{some rotation of } P$$
$$R(P^\perp) \subseteq P^\perp \text{ and } R|_{P^\perp} = \text{id}_{P^\perp}.$$ 

In other words, $R$ rotates the plane $P$ and leaves every vector of $P^\perp$ where it is.

Example: The transformation $R: \mathbb{R}^3 \to \mathbb{R}^3$ with (standard) matrix

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}$$

is a planar rotation in the $yz$-plane of $\mathbb{R}^3$.

Proposition 1: A planar rotation is an orthogonal transformation.

Proof: It suffices to check that $R: \mathbb{R}^n \to \mathbb{R}^n$ preserves lengths. For any $x \in \mathbb{R}^n$, consider the unique decomposition $x = p + w$ with $p \in P$ and $w \in P^\perp$. Then we have

$$\|Rx\|^2 = \|Rp + Rw\|^2 \text{ since } R \text{ is linear}$$
$$= \|Rp + w\|^2 \text{ since } R \text{ is the identity on } P^\perp$$
$$= \|Rp\|^2 + \|w\|^2 \text{ since } Rp \perp w$$
$$= \|p\|^2 + \|w\|^2 \text{ since } R|_P \text{ is a rotation in } P$$
$$= \|p + w\|^2 \text{ since } p \perp w$$
$$= \|x\|^2. \blacksquare$$

Proposition 2: A linear map $R: \mathbb{R}^n \to \mathbb{R}^n$ is a planar rotation if and only if there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ in which the matrix of $R$ is

$$\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & I
\end{bmatrix}.$$ (1)
**Proof:** (⇐) If there is such an orthonormal basis, then consider the plane \( P := \text{Span}\{v_1, v_2\} \). We have \( R(P) \subseteq P \) because the lower-left block is 0, and \( R|_P \) is a rotation of \( P \), because of the top-left block.

Moreover \( R \) satisfies \( Rv_i = v_i \) for \( i \geq 3 \) so that \( R \) is the identity on \( \text{Span}\{v_3, \ldots, v_n\} = P^\perp \). The last equality holds because the basis \( \{v_1, \ldots, v_n\} \) is orthogonal.

(⇒) Assume \( R \) is a planar rotation in a plane \( P \). Let \( \{v_1, v_2\} \) be an orthonormal basis of \( P \). Then we have

\[
\begin{align*}
  Pv_1 &= (\cos \theta)v_1 + (\sin \theta)v_2, \\
  Pv_2 &= (-\sin \theta)v_1 + (\cos \theta)v_2
\end{align*}
\]

for some angle \( \theta \).

Complete \( \{v_1, v_2\} \) to an orthonormal basis \( \{v_1, v_2, v_3, \ldots, v_n\} \) of \( \mathbb{R}^n \). Because \( v_3, \ldots, v_n \) are in \( P^\perp \), we have \( Rv_i = v_i \) for \( i \geq 3 \). Therefore \( R \) has matrix (1) in the basis \( \{v_1, \ldots, v_n\} \).

\( \blacksquare \)

## 2 Orthogonal matrices as rotations and reflections

The main theorems of section §6.5 are the following.

**Theorem 5.1.** Let \( A: \mathbb{R}^n \to \mathbb{R}^n \) be an orthogonal operator with \( \det A = 1 \). Then there is an orthonormal basis \( \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) in which the matrix of \( A \) is the block diagonal matrix

\[
\begin{bmatrix}
  R_{\theta_1} & & \\
  & R_{\theta_2} & \\
  & & \ddots \\
  & & & R_{\theta_k} \\
  & & & & I_{n-2k}
\end{bmatrix}
\]

where \( R_{\theta_j} \) is the 2-dimensional rotation

\[
R_{\theta_j} = \begin{bmatrix}
  \cos \theta_j & -\sin \theta_j \\
  \sin \theta_j & \cos \theta_j
\end{bmatrix}
\]

and \( I_{n-2k} \) denotes the \((n-2k) \times (n-2k)\) identity matrix.

**Theorem 5.2.** Let \( A: \mathbb{R}^n \to \mathbb{R}^n \) be an orthogonal operator with \( \det A = -1 \). Then there is an orthonormal basis \( \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) in which the matrix of \( A \) is the block diagonal matrix

\[
\begin{bmatrix}
  R_{\theta_1} & & \\
  & R_{\theta_2} & \\
  & & \ddots \\
  & & & R_{\theta_k} \\
  & & & & I_{n-2k-1}
\end{bmatrix}
\]

The theorems have the following geometric interpretation.

**Corollary of 5.1:** If \( A: \mathbb{R}^n \to \mathbb{R}^n \) is orthogonal with \( \det A = 1 \), then \( A \) is a product of at most \( \frac{\pi}{2} \) commuting planar rotations.
Proof: Let $Q$ be an orthogonal matrix satisfying $A = QBQ^{-1} = QBQ^T$ with

$$B = \begin{bmatrix} R_{\theta_1} & R_{\theta_2} & \cdots & R_{\theta_k} \\ I_2 & I_2 & \cdots & I_{n-2k} \\ & I_2 & \cdots & I_{n-2k} \\ & & \ddots & \ddots \\ & & & I_2 \end{bmatrix}.$$

The columns of $Q$ are the basis given by Thm 5.1. We can express $B$ as the product

$$B = \begin{bmatrix} R_{\theta_1} \\ I_2 \\ \vdots \\ I_{n-2k} \end{bmatrix} \begin{bmatrix} I_2 \\ R_{\theta_2} \\ \vdots \\ R_{\theta_k} \end{bmatrix} \begin{bmatrix} I_{n-2k} \\ I_{n-2k} \\ \vdots \\ I_{n-2k} \end{bmatrix} =: B_1B_2\ldots B_k$$

from which we obtain the factorization

$$A = QBQ^{-1} = (QB_1Q^{-1})(QB_2Q^{-1})\ldots(QB_kQ^{-1}) = A_1A_2\ldots A_k$$

in which each $A_i$ is a planar rotation, by Prop. 2. Also note that the $B_i$ commute with each other, and therefore so do the $A_i$. ■

Corollary of 5.2: If $A: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal with $\det A = -1$, then $A$ is a product of at most $\frac{n-1}{2}$ commuting planar rotations and a reflection which commutes with the rotations.

Proof: Let $Q$ be an orthogonal matrix satisfying $A = QBQ^{-1} = QBQ^T$ with

$$B = \begin{bmatrix} R_{\theta_1} & R_{\theta_2} & \cdots & R_{\theta_k} \\ I_2 & I_2 & \cdots & I_{n-2k} \\ & I_2 & \cdots & I_{n-2k} \\ & & \ddots & \ddots \\ & & & I_2 \end{bmatrix}.$$

The columns of $Q$ are the basis given by Thm 5.2. We can express $B$ as the product

$$B = \begin{bmatrix} R_{\theta_1} \\ I_2 \\ \vdots \\ I_{n-2k} \end{bmatrix} \begin{bmatrix} I_2 \\ R_{\theta_2} \\ \vdots \\ R_{\theta_k} \end{bmatrix} \begin{bmatrix} I_{n-2k} \\ I_{n-2k} \\ \vdots \\ I_{n-2k} \end{bmatrix} =: B_1B_2\ldots B_kB_{k+1}$$
from which we obtain the factorization

\[ A = Q B Q^{-1} \]

\[ = (Q B_1 Q^{-1})(Q B_2 Q^{-1}) \ldots (Q B_k Q^{-1})(Q B_{k+1} Q^{-1}) \]

\[ = A_1 A_2 \ldots A_k A_{k+1} \]

in which each \( A_i \) \((1 \leq i \leq k)\) is a planar rotation, whereas \( A_{k+1} \) is a reflection, which flips the vector \( v_n \). As before, the \( B_i \) commute with each other, and therefore so do the \( A_i \).

3 Examples

Here are a few simple examples.

Example: The transformation \( A: \mathbb{R}^3 \to \mathbb{R}^3 \) with matrix

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

is a rotation in the \( yz \)-plane composed with a reflection across the \( yz \)-plane (flipping the \( x \)-axis), and the two commute:

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

Example: The transformation \( A: \mathbb{R}^4 \to \mathbb{R}^4 \) with matrix

\[
A = \begin{bmatrix}
\cos \theta_1 & 0 & -\sin \theta_1 & 0 \\
0 & \cos \theta_2 & 0 & -\sin \theta_2 \\
\sin \theta_1 & 0 & \cos \theta_1 & 0 \\
0 & \sin \theta_2 & 0 & \cos \theta_2
\end{bmatrix}
\]

is a rotation in the \( x_1 x_3 \)-plane of \( \mathbb{R}^4 \) composed with a rotation in the \( x_2 x_4 \)-plane, and the two commute:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta_2 & 0 & -\sin \theta_2 \\
0 & 0 & 1 & 0 \\
0 & \sin \theta_2 & 0 & \cos \theta_2
\end{bmatrix}
\begin{bmatrix}
\cos \theta_1 & 0 & -\sin \theta_1 & 0 \\
0 & 1 & 0 & 0 \\
\sin \theta_1 & 0 & \cos \theta_1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta_2 & 0 & -\sin \theta_2 \\
0 & 0 & 1 & 0 \\
0 & \sin \theta_2 & 0 & \cos \theta_2
\end{bmatrix}
\]

Let us now illustrate the theorems and their proofs with a more substantial example.
Example: Consider the orthogonal matrix

\[ A = \frac{1}{6} \begin{bmatrix} 1 & -5 & -3 & 1 \\ 5 & -1 & 3 & -1 \\ 1 & 1 & -3 & -5 \\ 3 & 3 & -3 & 3 \end{bmatrix}. \]

Computing a diagonalization of \( A \) yields \( A = U D U^{-1} \) where

\[ D = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

is diagonal and

\[ U = [v \ v v_{3} \ v_{4}] = \begin{bmatrix} -2 + i & -2 - i & 1 & 0 \\ 1 + 2i & 1 - 2i & 0 & -1 \\ i & -i & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \]

has orthogonal columns (we dropped the normalization condition for simplicity).

Note that the eigenvalues \( \pm i \) can be written as \( e^{i\alpha} \) where \( \alpha \) happens to be \( \frac{\pi}{2} \). We will use the real and imaginary parts of the eigenvector \( v \) corresponding to the eigenvalue \( \lambda = i \). We have:

\[ \text{Re} v = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{Im} v = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \]

\[ A(\text{Re} v) = \text{Re}(\lambda v) = (\cos \alpha) \text{Re} v - (\sin \alpha) \text{Im} v = - \text{Im} v \]

\[ A(\text{Im} v) = \text{Im}(\lambda v) = (\sin \alpha) \text{Re} v + (\cos \alpha) \text{Im} v = \text{Re} v \]

so that in the (orthogonal) basis \( \{\text{Re} v, \text{Im} v, v_{3}, v_{4}\} \) the transformation \( A \) has matrix

\[ B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \]

Writing \( Q = [\text{Re} v \ \text{Im} v \ v_{3} \ v_{4}] \), we obtain a factorization

\[ A = Q B Q^{-1} \]

\[ = (Q \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Q^{-1})(Q \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} Q^{-1}) \]

\[ = A_{1}A_{2} \]
where $A_1$ is a planar rotation in the plane $\text{Span}\{\text{Re} \, v, \text{Im} \, v\} = \text{Span}\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\}$ and $A_2$ is the reflection which flips the vector $v_4 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$.