Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Orthogonal matrices and rotations

1 Planar rotations

Definition: A **planar rotation** in $\mathbb{R}^n$ is a linear map $R\colon \mathbb{R}^n \to \mathbb{R}^n$ such that there is a plane $P \subseteq \mathbb{R}^n$ (through the origin) satisfying

$$R(P) \subseteq P \text{ and } R|_P = \text{ some rotation of } P$$
$$R(P^\perp) \subseteq P^\perp \text{ and } R|_{P^\perp} = \mathrm{id}_{P^\perp}\,.$$

In other words, R rotates the plane P and leaves every vector of $P^\perp$ where it is.

Example: The transformation $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ with (standard) matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

is a planar rotation in the yz-plane of $\mathbb{R}^3$.

Proposition 1: A planar rotation is an orthogonal transformation.

Proof: It suffices to check that $R\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves lengths. For any $x \in \mathbb{R}^n$, consider the unique decomposition $x = p + w$ with $p \in P$ and $w \in P^\perp$. Then we have

$$\begin{aligned} \|Rx\|^2 &= \|Rp + Rw\|^2 \text{ since } R \text{ is linear} \\ &= \|Rp + w\|^2 \text{ since } R \text{ is the identity on } P^\perp \\ &= \|Rp\|^2 + \|w\|^2 \text{ since } Rp \perp w \\ &= \|p\|^2 + \|w\|^2 \text{ since } R|_P \text{ is a rotation in } P \\ &= \|p + w\|^2 \text{ since } p \perp w \\ &= \|x\|^2. \blacksquare \end{aligned}$$

Proposition 2: A linear map $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is a planar rotation if and only if there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ in which the matrix of R is

$$\left[\begin{array}{cc|c} \cos\theta & -\sin\theta & \mathbf{0} \\ \sin\theta & \cos\theta & \\ \hline \mathbf{0} & & I \end{array}\right]. \tag{1}$$

Proof: ($\Leftarrow$) If there is such an orthonormal basis, then consider the plane $P := \text{Span}\{v_1, v_2\}$. We have $R(P) \subseteq P$ because the lower-left block is 0, and $R|_P$ is a rotation of P, because of the top-left block.

Moreover R satisfies $Rv_i = v_i$ for $i \geq 3$ so that R is the identity on $\text{Span}\{v_3, \ldots v_n\} = P^\perp$. The last equality holds because the basis $\{v_1, \ldots, v_n\}$ is orthogonal.

($\Rightarrow$) Assume R is a planar rotation in a plane P. Let $\{v_1, v_2\}$ be an orthonormal basis of P. Then we have

$$Pv_1 = (\cos\theta)v_1 + (\sin\theta)v_2, \; Pv_2 = (-\sin\theta)v_1 + (\cos\theta)v_2$$

for some angle θ.

Complete $\{v_1, v_2\}$ to an orthonormal basis $\{v_1, v_2, v_3, \ldots, v_n\}$ of $\mathbb{R}^n$. Because $v_3, \ldots, v_n$ are in $P^\perp$, we have $Rv_i = v_i$ for $i \geq 3$. Therefore R has matrix (1) in the basis $\{v_1, \ldots, v_n\}$. ■

2 Orthogonal matrices as rotations and reflections

The main theorems of section §6.5 are the following.

Theorem 5.1. Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal operator with $\det A = 1$. Then there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ in which the matrix of A is the block diagonal matrix

$$\begin{bmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \ddots & & \\ & & & R_{\theta_k} & \\ & & & & I_{n-2k} \end{bmatrix}$$

where R_{θ_j} is the 2-dimensional rotation

$$R_{\theta_j} = \begin{bmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{bmatrix}$$

and I_{n-2k} denotes the $(n-2k) \times (n-2k)$ identity matrix.

Theorem 5.2. Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal operator with $\det A = -1$. Then there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ in which the matrix of A is the block diagonal matrix

$$\begin{bmatrix} R_{\theta_1} & & & & & \\ & R_{\theta_2} & & & & \\ & & \ddots & & & \\ & & & R_{\theta_k} & & \\ & & & & I_{n-2k-1} & \\ & & & & & -1 \end{bmatrix}.$$

The theorems have the following geometric interpretation.

Corollary of 5.1: If $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal with $\det A = 1$, then A is a product of at most $\frac{n}{2}$ commuting planar rotations.

Proof: Let Q be an orthogonal matrix satisfying $A = QBQ^{-1} = QBQ^T$ with

$$B = \begin{bmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \ddots & & \\ & & & R_{\theta_k} & \\ & & & & I_{n-2k} \end{bmatrix}.$$

The columns of Q are the basis given by Thm 5.1. We can express B as the product

$$\begin{aligned} B &= \begin{bmatrix} R_{\theta_1} & & & & \\ & I_2 & & & \\ & & \ddots & & \\ & & & I_2 & \\ & & & & I_{n-2k} \end{bmatrix} \begin{bmatrix} I_2 & & & & \\ & R_{\theta_2} & & & \\ & & \ddots & & \\ & & & I_2 & \\ & & & & I_{n-2k} \end{bmatrix} \cdots \begin{bmatrix} I_2 & & & & \\ & I_2 & & & \\ & & \ddots & & \\ & & & R_{\theta_k} & \\ & & & & I_{n-2k} \end{bmatrix} \\ &=: B_1 B_2 \ldots B_k \end{aligned}$$

from which we obtain the factorization

$$\begin{aligned} A &= QBQ^{-1} \\ &= (QB_1Q^{-1})(QB_2Q^{-1}) \ldots (QB_kQ^{-1}) \\ &= A_1 A_2 \ldots A_k \end{aligned}$$

in which each A_i is a planar rotation, by Prop. 2. Also note that the B_i commute with each other, and therefore so do the A_i. ■

Corollary of 5.2: If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal with $\det A = -1$, then A is a product of at most $\frac{n-1}{2}$ commuting planar rotations and a **reflection** which commutes with the rotations.

Proof: Let Q be an orthogonal matrix satisfying $A = QBQ^{-1} = QBQ^T$ with

$$B = \begin{bmatrix} R_{\theta_1} & & & & & \\ & R_{\theta_2} & & & & \\ & & \ddots & & & \\ & & & R_{\theta_k} & & \\ & & & & I_{n-2k-1} & \\ & & & & & -1 \end{bmatrix}.$$

The columns of Q are the basis given by Thm 5.2. We can express B as the product

$$\begin{aligned} B &= \begin{bmatrix} R_{\theta_1} & & & & \\ & I_2 & & & \\ & & \ddots & & \\ & & & I_2 & \\ & & & & I_{n-2k} \end{bmatrix} \cdots \begin{bmatrix} I_2 & & & & \\ & I_2 & & & \\ & & \ddots & & \\ & & & R_{\theta_k} & \\ & & & & I_{n-2k} \end{bmatrix} \begin{bmatrix} I_2 & & & & & \\ & I_2 & & & & \\ & & \ddots & & & \\ & & & I_2 & & \\ & & & & I_{n-2k-1} & \\ & & & & & -1 \end{bmatrix} \\ &=: B_1 B_2 \ldots B_k B_{k+1} \end{aligned}$$

from which we obtain the factorization

$$\begin{aligned}
A &= QBQ^{-1} \\
&= (QB_1Q^{-1})(QB_2Q^{-1})\dots(QB_kQ^{-1})(QB_{k+1}Q^{-1}) \\
&= A_1A_2\dots A_kA_{k+1}
\end{aligned}$$

in which each A_i $(1 \le i \le k)$ is a planar rotation, whereas A_{k+1} is a reflection, which flips the vector v_n. As before, the B_i commute with each other, and therefore so do the A_i. ■

3 Examples

Here are a few simple examples.

Example: The transformation $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ with matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

is a rotation in the yz-plane composed with a reflection across the yz-plane (flipping the x-axis), and the two commute:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: The transformation $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ with matrix

$$A = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

is a rotation in the x_1x_3-plane of $\mathbb{R}^4$ composed with a rotation in the x_2x_4-plane, and the two commute:

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}.
\end{aligned}$$

Let us now illustrate the theorems and their proofs with a more substantial example.

Example: Consider the orthogonal matrix

$$A = \frac{1}{6}\begin{bmatrix} 1 & -5 & -3 & 1 \\ 5 & -1 & 3 & -1 \\ 1 & 1 & -3 & -5 \\ 3 & 3 & -3 & 3 \end{bmatrix}.$$

Computing a diagonalization of A yields $A = UDU^{-1}$ where

$$D = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is diagonal and

$$U = \begin{bmatrix} v & \overline{v} & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} -2+i & -2-i & 1 & 0 \\ 1+2i & 1-2i & 0 & -1 \\ i & -i & -1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

has orthogonal columns (we dropped the normalization condition for simplicity).

Note that the eigenvalues $\pm i$ can be written as $e^{i\alpha}$ where α happens to be $\frac{\pi}{2}$. We will use the real and imaginary parts of the eigenvector v corresponding to the eigenvalue $\lambda = i$. We have:

$$\operatorname{Re} v = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ \operatorname{Im} v = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$A(\operatorname{Re} v) = \operatorname{Re}(\lambda v) = (\cos\alpha)\operatorname{Re} v - (\sin\alpha)\operatorname{Im} v = -\operatorname{Im} v$$

$$A(\operatorname{Im} v) = \operatorname{Im}(\lambda v) = (\sin\alpha)\operatorname{Re} v + (\cos\alpha)\operatorname{Im} v = \operatorname{Re} v$$

so that in the (orthogonal) basis $\{\operatorname{Re} v, \operatorname{Im} v, v_3, v_4\}$ the transformation A has matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Writing $Q = \begin{bmatrix} \operatorname{Re} v & \operatorname{Im} v & v_3 & v_4 \end{bmatrix}$, we obtain a factorization

$$\begin{aligned} A &= QBQ^{-1} \\ &= (Q \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Q^{-1})(Q \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} Q^{-1}) \\ &= A_1 A_2 \end{aligned}$$

where A_1 is a planar rotation in the plane $\mathrm{Span}\{\mathrm{Re}\, v, \mathrm{Im}\, v\} = \mathrm{Span}\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\}$ and A_2 is the reflection which flips the vector $v_4 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$.