Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Practice midterm 3

Name: ____

- This is a practice exam. The real exam will consist of 4 problems.
- In the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

1. __________/15
2. __________/15
3. __________/15
4. __________/10
5. __________/10
6. __________/10

Total: __________/75

Section 4.2

Notation: Given an $n \times n$ matrix A, let us denote by $\mu_{\text{alg}}(\lambda)$ and $\mu_{\text{geo}}(\lambda)$ the algebraic and geometric multiplicities, respectively, of an eigenvalue λ of A. In fact, both notions make sense for any scalar λ: both are zero when λ is not an eigenvalue of A.

Problem 1a. (5 pts) Let A be an $n \times n$ matrix. Prove the inequality

$$\operatorname{rank} A \geq \sum_{\lambda \neq 0} \mu_{\text{alg}}(\lambda). \tag{1}$$

b. (5 pts) When A is diagonalizable, prove that (1) is in fact an equality.

c. (5 pts) When A is non-diagonalizable, prove that we cannot conclude (in general) whether (1) is an equality or a strict inequality.

In other words, provide an example of non-diagonalizable matrix A such that (1) is an equality and an example of non-diagonalizable matrix B such that (1) is a strict inequality.

Section 5.1

Problem 2. Consider the complex vector space $\mathbb{C}^{n\times n}$ of complex $n \times n$ matrices. One of the following two formulas defines a complex inner product on $\mathbb{C}^{n\times n}$:

1. $(A, B) = \mathrm{tr}(A\overline{B})$
2. $(A, B) = \mathrm{tr}(AB^*)$

where $\mathrm{tr}\, M := \sum_{i=1}^{n} M_{ii}$ denotes the trace of a square matrix (sum of the diagonal entries) and $B^* := \overline{B}^T$ denotes the conjugate transpose.

a. (10 pts) Which formula is an inner product? Prove your answer.

b. (5 pts) Prove that the other formula is **not** an inner product.

Sections 5.2-5.3

Problem 3a. (6 pts) Consider the vectors $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$. Find the projection matrix onto the plane Span$\{a_1, a_2\}$ (directly, without using part b).

b. (5 pts) Consider the vector $v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$. Find the projection matrix onto the line Span$\{v\}$ (directly, without using part a).

c. (3 pts) Show $\text{Span}\{v\} = \text{Span}\{a_1, a_2\}^\perp$.

d. (1 pt) Check that the matrices you obtained in parts (a) and (b) add up to I, the 3×3 identity matrix.

Moral: Since projections onto a subspace E or its orthogonal complement $E^\perp$ determine each other via the relation $\text{Proj}_E + \text{Proj}_{E^\perp} = I$, we can compute one or the other, whichever is easier. The one with the smaller dimension is generally easier, e.g. part (b) was easier than (a).

Sections 5.4

Problem 4. (10 pts) Find the least squares fit of the form $y(x) = c + d(2^x)$ (for some $c, d \in \mathbb{R}$) through the data points $(x_i, y_i) = (0, 2); (1, 3); (2, 6)$.

Sections 5.5-5.6

Problem 5. Let V be a complex inner product space, and consider linear maps $A\colon V \to V$.

For each statement below, say if the statement is true or false (i.e. always true or not always true). Prove your answer.

a. (2.5 pts) If A and B are self-adjoint, then $A + B$ is also self-adjoint.

b. (2.5 pts) If A and B are self-adjoint, then AB is also self-adjoint.

c. (2.5 pts) If A and B are unitary, then $A + B$ is also unitary.

d. (2.5 pts) If A and B are unitary, then AB is also unitary.

Sections 6.2

Problem 6a. (2 pts) Which of the following matrices are diagonalizable by an orthogonal (i.e. unitary and real) matrix? **Circle** the answer(s) and **explain**.

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

b. (8 pts) Let A be a matrix you selected in part (a). Write a diagonalization of A by an **orthogonal** matrix U.