Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Practice midterm 2

Name: ______________________________

- This is a (long) practice exam. The real exam will consist of 4 problems.
- In the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

1. __________/15
2. __________/10
3. __________/10
4. __________/10
5. __________/5
6. __________/10
7. __________/15
8. __________/10
9. __________/10
10. __________/10
11. __________/10
12. __________/5

Total: __________/120

Section 2.5

Problem 1. Let A be an $m \times n$ matrix.

a. (5 pts) Show that A has linearly independent columns if and only if $A\colon \mathbb{R}^n \to \mathbb{R}^m$ preserves linear independence, in the following sense: For any collection of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ we have

$$\{v_1, \ldots, v_k\} \text{ is linearly independent } \Rightarrow \{Av_1, \ldots, Av_k\} \text{ is linearly independent.}$$

b. (5 pts) Show that $A\colon \mathbb{R}^n \to \mathbb{R}^m$ preserves linear independence if and only if for every subspace $S \subseteq \mathbb{R}^n$ we have $\dim AS = \dim S$.

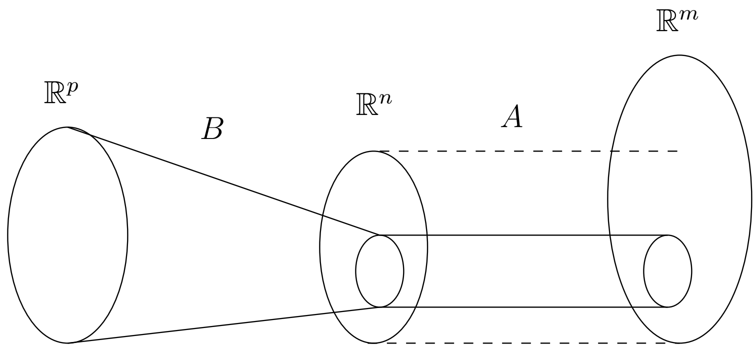

Figure 1: Composite AB.

c. (5 pts) Assume A has linearly independent columns. Let B be an $n \times p$ matrix. Show $\operatorname{rank} AB = \operatorname{rank} B$. (See figure 1.)

Section 2.6

Problem 2. (10 pts) Show that the general solution of the system

$$[A \mid b] = \left[\begin{array}{cccc|c} 3 & -1 & -1 & 9 & 1 \\ 2 & 1 & 6 & 1 & 9 \\ -3 & 2 & 5 & -12 & 4 \end{array}\right]$$

is $\{\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -8 \\ 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -11 \\ 0 \\ 3 \\ 4 \end{bmatrix} \mid s, t \in \mathbb{R}\}$.

Section 2.7

Problem 3. (10 pts) Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \end{bmatrix}$. Find a basis of each of the four fundamental subspaces of A, that is $\operatorname{Col} A$, $\operatorname{Null} A$, $\operatorname{Row} A$, $\operatorname{Null}(A^T)$.

Problem 4. Previously, we have shown that row operations preserve the null space and do **not** preserve the column space (in general).

a. (5 pts) Do row operations preserve the row space? Prove your answer.

b. (5 pts) Do row operations preserve the left null space? Prove your answer.

Problem 5. (5 pts) Let A, B be $m \times n$ matrices satisfying $\operatorname{Null} A = \operatorname{Null} B$ and $\operatorname{Col} A = \operatorname{Col} B$. Can we conclude $A = B$? Prove your answer.

Section 2.8

Problem 6. Consider the bases $\{u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\}$ and $\{v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}\}$ of $\mathbb{R}^2$.

a. (8 pts) Find the transition matrix from the basis $\{u_1, u_2\}$ to the basis $\{v_1, v_2\}$.

b. (2 pts) Find the coordinates of $5u_1 - 2u_2$ in the basis $\{v_1, v_2\}$.

Problem 7. Consider the "integration" map

$$T\colon P_1 \to P_2$$
$$p(x) \mapsto (Tp)(x) := \int_0^x p(t)\,\mathrm{dt}$$

For example, the polynomial $6x$ is sent to $\int_0^x 6t\,\mathrm{dt} = [3t^2]_0^x = 3x^2$. Note that T is linear.

a. (5 pts) Find the matrix representing T in the monomial bases $\{1, x\}$ of P_1 and $\{1, x, x^2\}$ of P_2.

b. (8 pts) Find the matrix representing T in the bases $\{2x+1, x-4\}$ of P_1 and $\{1, x-1, (x-1)^2\}$ of P_2.

c. (2 pts) Find the coordinates of $T\,(5(2x+1)+3(x-4))$ in the basis $\{1, x-1, (x-1)^2\}$ of P_2.

Problem 8. (10 pts) Let A be a 2×2 matrix which has an eigenvalue $\lambda_1 = 1$ with corresponding eigenvector $v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and an eigenvalue $\lambda_2 = 4$ with corresponding eigenvector $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find A.

Chapter 3

Problem 9. (10 pts) Compute the determinant

$$\begin{vmatrix} 0 & 0 & 0 & 5 \\ 23 & 11 & 3 & -7 \\ 2 & -15 & 0 & 8 \\ 0 & a & 0 & -5 \end{vmatrix}.$$

The answer may depend on the parameter $a \in \mathbb{R}$.

Section 4.1

Problem 10. (10 pts) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & -2 \\ 16 & -5 \end{bmatrix}.$$

Problem 11. (10 pts) (#4.1.6) A linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is called **nilpotent** if $A^k = \mathbf{0}$ for some $k \geq 1$. Show that if A is nilpotent, then 0 is the only eigenvalue of A.

Problem 12. (5 pts) Let A be a 2×2 matrix with the eigenvalue 7, of geometric multiplicty 2. Find A.