# Math 416 - Abstract Linear Algebra
# Fall 2011, section E1
# Some properties of matrices

Let $A$ be an $m \times n$ matrix. We discuss two interesting properties that $A$ can have, which will be stated in several equivalent ways.

## Property 1

**Proposition:** The following are equivalent.

1. The columns of $A$ are linearly independent.
2. The system $Ax = 0$ has the unique solution $x = 0$.
3. The null space of $A$ is trivial: $\operatorname{Null} A = \{0\} \subset \mathbb{R}^n$.
4. The system $Ax = b$ has at most one solution for any $b \in \mathbb{R}^m$.
5. The echelon form of $A$ has a pivot in every **column**.
6. $A$ has full column rank, i.e. $\dim \operatorname{Col} A = n$.
7. $A \colon \mathbb{R}^n \to \mathbb{R}^m$ is injective[1].
8. $A$ is left invertible.

## Property 2

**Proposition:** The following are equivalent.

1. The columns of $A$ span $\mathbb{R}^m$.
2. The system $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
3. $\operatorname{Col} A = \mathbb{R}^m$.
4. The echelon form of $A$ has a pivot in every **row**.
5. $A$ has full row rank, i.e. $\dim \operatorname{Row} A = m$.
6. $A \colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective[2].
7. $A$ is right invertible.

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[1] A function $f \colon X \to Y$ is **injective** if it satisfies $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$; equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. In other words, $f$ doesn't collapse points together, $f$ is an "inclusion" of a small set $X$ into a big set $Y$, $f$ doesn't lose information.

[2] A function $f \colon X \to Y$ is **surjective** if for every $y \in Y$, there is an $x \in X$ satisfying $f(x) = y$. In other words, $f$ "hits everything", $f$ sends a big set $X$ onto a small set $Y$, the image of $f$ is the whole target $Y$.

| | Property 1 | Property 2 |
|---|---|---|
| $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | Yes | Yes |
| $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ | Yes | No |
| $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ | No | Yes |
| $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ | No | No |

- When $m < n$ (more columns than rows), property 1 cannot hold.
- When $m = n$ (square matrix), property 1 is equivalent to property 2.
- When $m > n$ (more rows than columns), property 2 cannot hold.
- That is all we can say in general.

**Remark:** Only left invertibility and right invertibility require a bit of work to show (and are arguably less important). Feel free to ignore them. The other equivalences in the propositions are straightforward and important to understand. It is a good exercise to work them out.

We can also combine properties 1 and 2. As noted above, this can only happen for **square** matrices ($m = n$).

**Proposition:** The following are equivalent.

1. The columns of $A$ form a basis of $\mathbb{R}^m$.
2. The system $Ax = b$ has a unique solution for any $b \in \mathbb{R}^m$.
3. $\operatorname{Null} A = \{0\}$ and $\operatorname{Col} A = \mathbb{R}^m$.
4. The echelon form of $A$ has a pivot in every row and every column.
5. The reduced echelon form of $A$ is the identity matrix $I$.
6. $A$ is a product of elementary matrices.
7. $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is bijective[3].
8. $A$ is invertible.
9. $A$ is square and satisfies property 1.
10. $A$ is square and satisfies property 2.
11. $\det A \neq 0$.

[3]A function $f\colon X \to Y$ is **bijective** if it is both injective and surjective. In other words, for every $y \in Y$, there is a unique $x \in X$ satisfying $f(x) = y$.