Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Gram-Schmidt orthogonalization

Let us illustrate the fact that the Gram-Schmidt orthogonalization process works in any inner product space, not just $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Example: Consider the real inner product space $C[0,1] := \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous } \}$ with its usual inner product

$$(f,g) = \int_0^1 f(t)g(t)\,\mathrm{dt}.$$

Apply Gram-Schmidt to the linearly independent collection $\{v_1 = 1, v_2 = t\}$.

Solution: $u_1 = \frac{v_1}{\|v_1\|}$

$$\|v_1\|^2 = (v_1, v_1) = (1,1) = \int_0^1 (1)(1)\,\mathrm{dt} = 1$$

$\Rightarrow u_1 = \frac{v_1}{\sqrt{1}} = v_1 = 1$

$v_2^\perp = v_2 - \mathrm{Proj}_{u_1}(v_2) = v_2 - (v_2, u_1)u_1$

$$(v_2, u_1) = (t, 1) = \int_0^1 (t)(1)\,\mathrm{dt} = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$\Rightarrow v_2^\perp = v_2 - \frac{1}{2}u_1 = t - \frac{1}{2}$

$u_2 = \frac{v_2^\perp}{\|v_2^\perp\|}$

$$\|v_2^\perp\|^2 = (v_2^\perp, v_2^\perp) = (t - \frac{1}{2}, t - \frac{1}{2}) = (t,t) - (t,1) + \frac{1}{4}(1,1) = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$\Rightarrow u_2 = \frac{v_2^\perp}{1/\sqrt{12}} = \sqrt{12}(t - \frac{1}{2}) = \sqrt{3}(2t-1)$

The resulting **orthonormal** basis of $\mathrm{Span}\{1, t\}$ is $\{1, \sqrt{3}(2t-1)\}$.

Remark: If we only want orthogonal vectors, without caring about their norms, then the algorithm outputs the **orthogonal** basis $\{1, t - \frac{1}{2}\}$.