Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Practice Final

Name: ____

- This is a (long) practice exam. The real exam will consist of 6 problems.
- In the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

1. __________/10
2. __________/15
3. __________/15
4. __________/10
5. __________/10
6. __________/10
7. __________/10
8. __________/10
9. __________/10
10. __________/15
11. __________/10
12. __________/10

Total: __________/135

Problem 1. (10 pts) Let V, W be finite-dimensional vector spaces. Show that V is isomorphic to W if and only if V and W have the same dimension.

Problem 2a. (6 pts) Show that vectors $v_1, \ldots, v_n \in V$ are linearly independent if and only if there is a linear map $T\colon V \to W$ such that $Tv_1, \ldots, Tv_n$ are linearly independent. [The content is in the "if" direction.]

b. (4 pts) Let $X := (a, b) \subseteq \mathbb{R}$ be an open interval of the real line, and consider the vector space of smooth functions on X

$$C^\infty(X) := \{f \colon X \to \mathbb{R} \mid \text{ derivatives } f^{(n)} \text{ exist for all } n \geq 1\}.$$

Let $x \in X$ be some point in the interval X. Check that

$$T_x \colon C^\infty(X) \to \mathbb{R}^2$$
$$f \mapsto \begin{bmatrix} f(x) \\ f'(x) \end{bmatrix}$$

is a linear map.

c. (5 pts) Using parts (a) and (b), prove the following trick used in differential equations. Let $f, g\colon X \to \mathbb{R}$ be smooth functions. If there is a point $x \in X$ such that the determinant (called the "Wronskian")

$$\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

is non-zero, then the functions $f, g \in C^\infty(X)$ are linearly independent.

Problem 3. Let $\mathcal{A} := \{v_1, \ldots, v_n\}$ be a basis of V and $\varphi_{\mathcal{A}} \colon V \xrightarrow{\simeq} \mathbb{R}^n$ the isomorphism giving the coordinates with respect to $\mathcal{A}$, i.e. $\varphi_{\mathcal{A}}(v) := [v]_{\mathcal{A}}$.

Let $\mathcal{B} := \{w_1, \ldots, w_m\}$ be a basis of W. Let $T \colon V \to W$ be a linear map and let $[T]_{\mathcal{B}\mathcal{A}}$ be the $m \times n$ matrix representing T with respect to the bases $\mathcal{A}$ and $\mathcal{B}$, as in the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi_{\mathcal{A}} \downarrow \simeq & & \simeq \downarrow \varphi_{\mathcal{B}} \\ \mathbb{R}^n & \xrightarrow[{[T]_{\mathcal{B}\mathcal{A}}}]{} & \mathbb{R}^m. \end{array}$$

a. (5 pts) Show $\varphi_{\mathcal{A}}(\ker T) = \mathrm{Null}[T]_{\mathcal{B}\mathcal{A}}$. In words: "the null space is the expression of the kernel in coordinates".

b. (5 pts) In the same setup, show $\varphi_{\mathcal{B}}(\operatorname{im} T) = \operatorname{Col}[T]_{\mathcal{B}\mathcal{A}}$. In words: "the column space is the expression of the image in coordinates".

c. (5 pts) Let $T\colon V \to W$ be a linear map, $\{v_1, v_2, v_3\}$ a basis of V, and $\{w_1, w_2\}$ a basis of W such that the matrix representing T with respect to those bases is

$$[T]_{\{w_i\}\{v_i\}} = \begin{bmatrix} 3 & 9 & -6 \\ 1 & 3 & -2 \end{bmatrix}.$$

Find a basis of $\ker T \subset V$ and a basis of $\operatorname{im} T \subset W$.

Problem 4. (10 pts) Consider the "Fibonacci-like" sequence given by

$$c_0 = 0$$
$$c_1 = 1$$
$$c_n = c_{n-1} + 2c_{n-2} \text{ for } n \geq 2.$$

For example, the first few numbers c_n are $0, 1, 1, 3, 5, 11, 21, 43, \ldots$

Find c_{500}.

Problem 5. (10 pts) Let A be an operator with **distinct** eigenvalues (i.e. all eigenvalues have algebraic multiplicity 1), and B an operator that commutes with A. Show that B is diagonalizable, moreover with the same eigenvectors as A.

Problem 6. (10 pts) Let V, W be n-dimensional complex inner product spaces. Show that V and W are isometrically isomorphic.

Problem 7. (10 pts) Let V be a finite-dimensional complex vector space and $\{v_1, \ldots, v_n\}$ a basis of V. For any vectors $v, w \in V$, express them in the given basis

$$v = \sum_i \alpha_i v_i, \ w = \sum_i \beta_i v_i$$

and define

$$(v, w) := \sum_i \alpha_i \overline{\beta_i} \in \mathbb{C}.$$

Show that this formula (v, w) defines a complex inner product on V.

Problem 8. (10 pts) Consider the plane P in $\mathbb{R}^3$ given by the equation

$$2x_1 - 4x_2 + x_3 = 1.$$

Note that the plane does **not** go through the origin.

Find the distance from the point $\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ to the plane.

Problem 9. (10 pts) Consider the real inner product space

$$C[-1,1] := \{f\colon [-1,1] \to \mathbb{R} \mid f \text{ is continuous } \}$$

with its usual inner product

$$(f,g) = \int_{-1}^{1} f(t)g(t)\,\mathrm{dt}.$$

Consider the absolute value function $f(t) = |t|$. Find $\mathrm{Proj}_{\{1,t,t^2\}}(f)$.

Problem 10. Let V be a finite-dimensional complex inner product space.

a. (5 pts) Show that a normal operator $A\colon V \to V$ with **real** eigenvalues is self-adjoint.

b. (10 pts) Using part (a), show that any normal operator $N\colon V \to V$ is the product of a unitary operator and a self-adjoint operator that commute with each other. Hint: A normal operator whose eigenvalues have modulus 1 is unitary.

Problem 11. Consider the set of all invertible $n \times n$ (real) matrices

$$GL(n) := \{A\colon \mathbb{R}^n \to \mathbb{R}^n \mid A \text{ is linear and invertible } \}.$$

Consider the function $\operatorname{sign}\colon GL(n) \to \{1, -1\}$ defined by

$$\operatorname{sign}(A) = \begin{cases} 1 & \text{if } A \text{ is orientation-preserving} \\ -1 & \text{if } A \text{ is orientation-reversing.} \end{cases}$$

a. (4 pts) Show that sign is compatible with products:

$$\operatorname{sign}(AB) = \operatorname{sign}(A) \operatorname{sign}(B).$$

b. (6 pts) Show that a permutation matrix P_σ is orientation-preserving if and only if σ is even. In other words, show the equality $\operatorname{sign}(P_\sigma) = \operatorname{sign}(\sigma)$.

Problem 12. Consider the quadratic form on $\mathbb{R}^2$ given by

$$Q(x) = 4x_1x_2 + 3x_2^2.$$

a. (2 pts) Find the unique symmetric matrix A satisfying $Q(x) = (Ax, x)$.

b. (8 pts) Orthogonally diagonalize A. In other words, find a factorization $A = UDU^T$, where U is orthogonal and D is diagonal.