Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Working in coordinates

In these notes, we explain the idea of working "in coordinates" or coordinate-free, and how the two are related.

1 Expressing vectors in coordinates

Let V be an n-dimensional vector space. Recall that a choice of basis $\{v_1, \ldots, v_n\}$ of V is the same data as an isomorphism $\varphi\colon V \simeq \mathbb{R}^n$, which sends the basis $\{v_1, \ldots, v_n\}$ of V to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. In other words, we have

$$\varphi\colon V \xrightarrow{\simeq} \mathbb{R}^n$$

$$v_i \mapsto e_i$$

$$v = c_1 v_1 + \ldots + c_n v_n \mapsto \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

This allows us to manipulate abstract vectors $v = c_1 v_1 + \ldots + c_n v_n$ simply as lists of numbers, the **coordinate** vectors $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ with respect to the basis $\{v_1, \ldots, v_n\}$. Note that the coordinates of $v \in V$ **depend** on the choice of basis.

Notation: Write $[v]_{\{v_i\}} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ for the coordinates of $v \in V$ with respect to the basis $\{v_1, \ldots, v_n\}$. For shorthand notation, let us name the basis $\mathcal{A} := \{v_1, \ldots, v_n\}$ and then write $[v]_{\mathcal{A}}$ for the coordinates of v with respect to the basis $\mathcal{A}$.

Example: Using the monomial basis $\{1, x, x^2\}$ of $P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\}$, we obtain an isomorphism

$$\varphi\colon P_2 \xrightarrow{\simeq} \mathbb{R}^3$$

$$a_0 + a_1 x + a_2 x^2 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

In the notation above, we have $[a_0 + a_1 x + a_2 x^2]_{\{x^i\}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Example: Let us use a different basis of P_2, the basis "expanded around 5" $\{1, x-5, (x-5)^2\}$. Then we have

$$\begin{aligned}
1 &= 1 \\
x &= (x-5) + 5 \\
&= 5(1) + 1(x-5) \\
x^2 &= (x-5)^2 + 10x - 25 \\
&= (x-5)^2 + 10(x-5) + 25 \\
&= 25(1) + 10(x-5) + 1(x-5)^2
\end{aligned}$$

so that in the notation above, the coordinates of the standard monomials are

$$[1]_{\{(x-5)^i\}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[x]_{\{(x-5)^i\}} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$[x^2]_{\{(x-5)^i\}} = \begin{bmatrix} 25 \\ 10 \\ 1 \end{bmatrix}.$$

2 Expressing transformations in coordinates

If vector spaces can be expressed in coordinates, then linear transformations between them can also be expressed in coordinates.

Theorem: Let V be an n-dimensional vector space, W an m-dimensional vector space, and $T\colon V \to W$ a linear transformation. Let $\mathcal{A} = \{v_1, \dots, v_n\}$ be a basis of V and $\mathcal{B} = \{w_1, \dots, w_m\}$ a basis of W. Then there is a unique $m \times n$ matrix A satisfying

$$[Tv]_{\mathcal{B}} = A[v]_{\mathcal{A}}$$

for all $v \in V$, given by the formula

$$A = \begin{bmatrix} [Tv_1]_{\mathcal{B}} & [Tv_2]_{\mathcal{B}} & \dots & [Tv_n]_{\mathcal{B}} \end{bmatrix}.$$

We denote this matrix $[T]_{\mathcal{BA}}$ and call it the matrix **representing** T with respect to the bases $\mathcal{A}$ and $\mathcal{B}$.

Slick proof: The choice of bases $\mathcal{A}$ and $\mathcal{B}$ defines isomorphisms $\varphi\colon V \simeq \mathbb{R}^n$ and $\psi\colon W \simeq \mathbb{R}^m$. There is a unique linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^m$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi \big\downarrow \simeq & & \simeq \big\downarrow \psi \\ \mathbb{R}^n & \xrightarrow[A]{} & \mathbb{R}^m \end{array}$$

commute, namely the transformation $A = \psi T \varphi^{-1}$. By §1.3.2, A corresponds to the $m \times n$ matrix A whose i^{th} column is $\psi T \varphi^{-1}(e_i) = \psi T v_i = [Tv_i]_{\mathcal{B}}$. ■

Direct proof: Since T is linear, it is determined by its values $Tv_1, Tv_2, \ldots, Tv_n$ on a basis. More precisely, for $v = c_1 v_1 + \ldots + c_n v_n$, we have

$$Tv = T(c_1 v_1 + \ldots + c_n v_n) = c_1 T v_1 + \ldots + c_n T v_n.$$

Taking coordinates with respect to the basis $\mathcal{B}$ of W, we obtain

$$\begin{aligned} [Tv]_{\mathcal{B}} &= c_1 [Tv_1]_{\mathcal{B}} + \ldots + c_n [Tv_n]_{\mathcal{B}} \\ &= \begin{bmatrix} [Tv_1]_{\mathcal{B}} & [Tv_2]_{\mathcal{B}} & \ldots & [Tv_n]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} [Tv_1]_{\mathcal{B}} & [Tv_2]_{\mathcal{B}} & \ldots & [Tv_n]_{\mathcal{B}} \end{bmatrix} [v]_{\mathcal{A}}. \ \blacksquare \end{aligned}$$

Example: Consider the differentiation operator $D\colon P_2 \to P_1$ which sends $p(x)$ to $p'(x)$. Let us find the matrix representing D with respect to the monomial bases $\{1, x, x^2\}$ of P_2 and $\{1, x\}$ of P_1. We have

$$\begin{aligned} D(1) &= 0 \\ D(x) &= 1 \\ D(x^2) &= 2x \end{aligned}$$

which have coordinate vectors

$$\begin{aligned} [D(1)]_{\{1,x\}} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [D(x)]_{\{1,x\}} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [D(x^2)]_{\{1,x\}} &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

so that D is represented by the matrix

$$[D]_{\{1,x\}\{1,x,x^2\}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Now keep the monomial basis $\{1, x, x^2\}$ in the source P_2 and take the basis expanded around 5 $\{1, x-5\}$ in the target P_1. Then we have

$$\begin{aligned} D(1) &= 0 \\ D(x) &= 1 \\ D(x^2) &= 2x = 2(x-5) + 10 = 10 + 2(x-5) \end{aligned}$$

which have coordinate vectors

$$\begin{aligned} [D(1)]_{\{1,x-5\}} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [D(x)]_{\{1,x-5\}} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [D(x^2)]_{\{1,x-5\}} &= \begin{bmatrix} 10 \\ 2 \end{bmatrix} \end{aligned}$$

so that D is represented by the matrix

$$[D]_{\{1,x-5\}\{1,x,x^2\}} = \begin{bmatrix} 0 & 1 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

Now take the basis expanded around 5 $\{1, x-5, (x-5)^2\}$ in the source P_2 and keep the monomial basis $\{1, x\}$ in the target P_1. Then we have

$$\begin{aligned} D(1) &= 0 \\ D(x-5) &= 1 \\ D(x-5)^2 &= 2(x-5) = -10 + 2x \end{aligned}$$

which have coordinate vectors

$$\begin{aligned} [D(1)]_{\{1,x\}} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [D(x-5)]_{\{1,x\}} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [D(x-5)^2]_{\{1,x\}} &= \begin{bmatrix} -10 \\ 2 \end{bmatrix} \end{aligned}$$

so that D is represented by the matrix

$$[D]_{\{1,x\}\{1,x-5,(x-5)^2\}} = \begin{bmatrix} 0 & 1 & -10 \\ 0 & 0 & 2 \end{bmatrix}.$$

Finally, take the bases expanded around 5 $\{1, x-5, (x-5)^2\}$ in the source P_2 and $\{1, x-5\}$ in the target P_1. Then we have

$$\begin{aligned} D(1) &= 0 \\ D(x-5) &= 1 \\ D(x-5)^2 &= 2(x-5) \end{aligned}$$

which have coordinate vectors

$$[D(1)]_{\{1,x-5\}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[D(x-5)]_{\{1,x-5\}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[D(x-5)^2]_{\{1,x-5\}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

so that D is represented by the matrix

$$[D]_{\{1,x-5\}\{1,x-5,(x-5)^2\}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

3 Switching between the two

The discussion above has two important consequences – as long as we don't mind going back and forth between coordinate world and coordinate-free world.

Upshot 1: The study of finite-dimensional vector spaces reduces to the study of spaces of the form $\mathbb{R}^n$, where we can compute everything explicitly.

Upshot 2: Every statement in coordinates has a coordinate-free analogue.

Let us illustrate this principle with the following example.

Theorem: Rank-nullity theorem, in coordinates. Let A be an $m \times n$ matrix. Then we have

$$\dim \operatorname{Null} A + \operatorname{rank} A = \dim \operatorname{Null} A + \dim \operatorname{Col} A = n.$$

Theorem: Rank-nullity theorem, coordinate-free. Let V be an n-dimensional vector space, W an m-dimensional vector space, and $T\colon V \to W$ a linear transformation. Then we have

$$\dim \ker T + \operatorname{rank} T = \dim \ker T + \dim \operatorname{im} T = \dim V = n.$$

Proof: Pick isomorphisms $\varphi\colon V \simeq \mathbb{R}^n$ and $\psi\colon W \simeq \mathbb{R}^m$, and let A be the matrix representing T in coordinates:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi \downarrow \simeq & & \simeq \downarrow \psi \\ \mathbb{R}^n & \xrightarrow[A]{} & \mathbb{R}^m. \end{array}$$

We have $\ker T = \varphi^{-1} \operatorname{Null} A$ which implies

$$\dim \ker T = \dim \left(\varphi^{-1} \operatorname{Null} A\right) = \dim \operatorname{Null} A$$

and also $\operatorname{im} T = \psi^{-1} \operatorname{Col} A$ which implies

$$\dim \operatorname{im} T = \dim \left(\psi^{-1} \operatorname{Col} A\right) = \dim \operatorname{Col} A.$$

Using the rank-nullity theorem in coordinates, we obtain

$$\dim \ker T + \dim \operatorname{im} T = \dim \operatorname{Null} A + \dim \operatorname{Col} A = n. \blacksquare$$