Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Column space and null space

The following example illustrates the notion of dimension and "culling down" a linearly dependent collection of vectors.

Let

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Find the dimension of $\operatorname{Col} A$ and $\operatorname{Null} A$, as well as a basis for each.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the pivots are in columns 1 and 3, we conclude $\dim \operatorname{Col} A = 2$ and a basis of $\operatorname{Col} A$ is given by $\{a_1, a_3\} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$.

Remark 1: We have culled down the linearly dependent collection $\{a_1, a_2, a_3\}$ to a basis of $\operatorname{Span}\{a_1, a_2, a_3\} = \operatorname{Col} A$. In other words, since a_2 is already in $\operatorname{Span}\{a_1, a_3\}$, we have $\operatorname{Span}\{a_1, a_2, a_3\} = \operatorname{Span}\{a_1, a_3\}$ and a_2 can be discarded.

Remark 2: Why does this algorithm work? Every column which does **not** have a pivot can be expressed as a linear combination of the pivot columns.

In our example, we have col 2 = 2 col 1, that is:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since row operations **preserve** linear dependence relations among columns, the same relation holds among the columns of the original matrix A. In other words, we have $a_2 = 2a_1$, that is:

$$\begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$$

For the null space, we are solving the system $Ax = \mathbf{0}$. Explicitly, row reduction gives

$$\begin{bmatrix} A \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Note that x_2 is the only free variable. The lead variables x_1, x_3 are determined by x_2 via the equations

$$\begin{aligned} x_1 + 2x_2 \qquad &= 0 \\ x_3 &= 0. \end{aligned}$$

Therefore the solution set is

$$\begin{aligned} \operatorname{Null} A &= \{ \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} \mid x_2 \in \mathbb{R} \} \\ &= \{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \mid x_2 \in \mathbb{R} \} \\ &= \operatorname{Span}\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}. \end{aligned}$$

In fact, $\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}$ is a basis of $\operatorname{Null} A$ and we conclude $\dim \operatorname{Null} A = 1$.

Remark 3: The fact that $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is in $\operatorname{Null} A$ is saying $-2a_1 + a_2 = \mathbf{0}$. In other words, it expresses the dependence relation $a_2 = 2a_1$ among the columns of A.

Remark 4: The algorithm to find a basis of $\operatorname{Null} A$ is to in turn set one of the free variables equal to 1 and the other free variables equal to 0, yielding what are sometimes called "special solutions" of the system $Ax = \mathbf{0}$. For example, consider

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_2, x_3 and the lead variable x_1 is determined via $x_1 + 5x_2 + 7x_3 = 0$. The solution set is

$$\begin{aligned} \operatorname{Null} A &= \{ \begin{bmatrix} -5x_2 - 7x_3 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \} \\ &= \{ x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \} \\ &= \operatorname{Span}\{ \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix} \}. \end{aligned}$$

Setting $x_2 = 1, x_3 = 0$ yields the special solution $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ whereas setting $x_2 = 0, x_3 = 1$ yields the special solution $\begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$. Together they form a basis of $\operatorname{Null} A$.

Remark 5: The discussion above implies

$$\begin{aligned} \dim \operatorname{Col} A &= \text{ number of columns with a pivot} \\ \dim \operatorname{Null} A &= \text{ number of columns without a pivot.} \end{aligned}$$

In particular, we have

$$\dim \operatorname{Col} A + \dim \operatorname{Null} A = \text{ number of columns } = n.$$