Math 415 - Applied Linear Algebra
Fall 2010, section D1
Practice Exam 3, Wednesday November 10

Name: _____

This is a practice exam to prepare for exam 3. It was calibrated to take around 80 minutes, which means the actual exam will be shorter.

Problem 1. Consider the ordered bases $\{u_1, u_2\}$ and $\{v_1, v_2\}$ of $\mathbb{R}^2$, where $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

a. Find the transition matrix S from $\{u_1, u_2\}$ to $\{v_1, v_2\}$.

b. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator whose matrix representation with respect to the basis $\{u_1, u_2\}$ is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find the matrix representation B of L with respect to the basis $\{v_1, v_2\}$.

c. Find $L(5v_1 + v_2)$, expressed in a basis of your choice.

Problem 2. Consider the plane in $\mathbb{R}^3$ containing the points $P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

a. Find a normal vector N for the plane.

b. Find an equation defining the plane.

c. Find the distance from the point $P = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ to the plane.

Problem 3. Let S be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

a. Find a basis of the orthogonal complement $S^\perp$, i.e. the plane orthogonal to the line S.

b. Is the vector $v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ orthogonal to S? Explain.

c. Find the orthogonal projection p of v onto S.

d. Find the unique decomposition of v as a sum of a vector in S and a vector in $S^\perp$.

Problem 4. You conduct an experiment and measure data points $(0, -1)$, $(1, 1)$, $(2, 2)$. You believe the quantity y depends on x as a polynomial $y(x) = c_0 + c_1 x$ of degree at most 1.

a. What linear system should the coefficients c_0, c_1 satisfy if it were indeed the case?

b. Find the best fit through the data, i.e. the least squares solution to the system in part (a).

Problem 5. Consider the vectors $v_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $v_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

a. Is $\{v_1, v_2\}$ an orthonormal set? Prove your answer.

b. Complete the set $\{v_1, v_2\}$ to an orthonormal basis of $\mathbb{R}^3$. In other words, find a vector $v_3 \in \mathbb{R}^3$ such that $\{v_1, v_2, v_3\}$ is an orthonormal set.

c. Find the coordinates of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with respect to the basis you found in part (b).

Problem 6. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator that first rotates $30°$ about the x-axis and then rotates $45°$ about the z-axis. (We use the right-hand sign convention.)

a. Let A be the standard matrix representation of L. Is A an orthogonal matrix? Explain.

b. Find the distance between $L(e_1)$ and $L(e_2)$.

Problem 7. Consider the vector space $C[1, 2]$ with inner product $\langle f, g \rangle = \int_1^2 f(x)g(x)\mathrm{d}x$.

a. The vectors 1 and x are linearly independent but not orthogonal. Using the Gram-Schmidt process, find an orthonormal basis of $\mathrm{Span}\{1, x\}$.

b. Find the orthogonal projection of x^2 onto $\mathrm{Span}\{1, x\}$.

Problem 8. Find the eigenvalues of $A = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ and corresponding eigenvectors.