Math 415 - Applied Linear Algebra

Diagonalization of symmetric matrices

Theorem: A real matrix A is symmetric *if and only if* A can be diagonalized by an orthogonal matrix, i.e. $A = UDU^{-1}$ with U orthogonal and D diagonal.

To illustrate the theorem, let us diagonalize the following matrix by an orthogonal matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Here is a shortcut to find the eigenvalues. Note that rows 2 and 3 are multiples of row 1, which means A has nullity 2, so that 0 is an eigenvalue with (algebraic) multiplicity at least 2. Moreover the sum of the three eigenvalues is $\operatorname{tr}(A) = 3$, so the third eigenvalue must be 3.

Let us find the eigenvectors:

$$\lambda_1 = \lambda_2 = 0 : A - 0I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Take $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. They form a basis of the 0-eigenspace, albeit not an orthonormal basis. Let us apply Gram-Schmidt to obtain an orthonormal basis. (We call the intermediate orthogonal vectors w_i.)

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$w_2 = v_2 - \operatorname{proj}_{u_1}(v_2) = v_2 - \langle u_1, v_2 \rangle u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}(1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\lambda_3 = 3 : A - 3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Take $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and normalize it:

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We conclude $A = UDU^{-1}$, where $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is orthogonal and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonal.

Trace of a matrix

Definition: The **trace** of an $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{tr}(A) = a_{1,1} + a_{2,2} + \ldots + a_{n,n}.$$

Examples: $\operatorname{tr}\left(\begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}\right) = 6,$ $\operatorname{tr}\left(\begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix}\right) = 9,$ $\operatorname{tr}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 15,$

$$\operatorname{tr}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 3, \qquad \operatorname{tr}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 4.$$

Theorem: For any two $n \times n$ matrices A and B, we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof:

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} \\ &= \sum_{k=1}^{n} (BA)_{kk} \\ &= \operatorname{tr}(BA). \quad \square \end{aligned}$$

Example:

$$\operatorname{tr}\left(\begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 55 & 3 \\ -4 & -4 \end{bmatrix}\right) = 51$$

$$\operatorname{tr}\left(\begin{bmatrix} 10 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 38 & 54 \\ 13 & 13 \end{bmatrix}\right) = 51.$$

Corollary: Similar matrices have the same trace.

Proof: Given $B = SAS^{-1}$, we have $\operatorname{tr}(B) = \operatorname{tr}(SAS^{-1}) = \operatorname{tr}(S^{-1}SA) = \operatorname{tr}(A)$. $\square$

Theorem: Let A be an $n \times n$ matrix. Then the sum of the eigenvalues of A (counted with multiplicity) is $\operatorname{tr}(A)$.

Proof: By Schur's theorem (6.4.3), A is similar to an upper triangular matrix, i.e. we have $A = STS^{-1}$ for some nonsingular matrix S and upper triangular matrix T (both of which might have complex entries). Recalling that similar matrices have the same eigenvalues – indeed, the same characteristic polynomial – we obtain:

$$\begin{aligned} \operatorname{tr}(A) &= \operatorname{tr}(T) \text{ , since } T \text{ is similar to } A \\ &= \text{sum of eigenvalues of } T \text{ , since } T \text{ is upper triangular} \\ &= \text{sum of eigenvalues of } A \text{ , since } A \text{ is similar to } T. \quad \square \end{aligned}$$

Example: Let us find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

The eigenvalues are 1 and 3. Their sum is 4, which is indeed $\operatorname{tr}(A) = 2 + 2$.