Math 285 - Intro Differential Equations
Spring 2011, sections G1 and X1
Fourier series solutions

Section 9.4

14. The equation is $2x'' + 0.1x' + 18x = F(t) = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$. Let us find the steady 2π-periodic solution:

$$x_{sp}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$x'_{sp}(t) = +\sum_{n=1}^{\infty} -na_n \sin nt + nb_n \cos nt$$

$$x''_{sp}(t) = +\sum_{n=1}^{\infty} -n^2 a_n \cos nt - n^2 b_n \sin nt$$

$$\begin{aligned} 2x''_{sp} + 0.1x'_{sp} + 18x_{sp} &= 9a_0 + \sum_{n=1}^{\infty} [(-2n^2 a_n + 0.1nb_n + 18a_n)\cos nt + \\ &\quad + (-2n^2 b_n - 0.1na_n + 18b_n)\sin nt] \\ &= 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt. \end{aligned}$$

Equating the coefficients of corresponding terms, we obtain the equations:

$$\begin{aligned} 9a_0 &= 0 \\ (18 - 2n^2)a_n + 0.1n\, b_n &= 0 \\ -0.1n\, a_n + (18 - 2n^2)b_n &= \frac{4(-1)^{n+1}}{n} \end{aligned}$$

which can be written in matrix form:

$$\begin{bmatrix} 18 - 2n^2 & 0.1n \\ -0.1n & 18 - 2n^2 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4(-1)^{n+1}}{n} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} a_n \\ b_n \end{bmatrix} &= \begin{bmatrix} 18 - 2n^2 & 0.1n \\ -0.1n & 18 - 2n^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{4(-1)^{n+1}}{n} \end{bmatrix} \\ &= \frac{1}{(18 - 2n^2)^2 + 0.01n^2} \begin{bmatrix} 18 - 2n^2 & -0.1n \\ 0.1n & 18 - 2n^2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{4(-1)^{n+1}}{n} \end{bmatrix} \\ &= \frac{1}{(18 - 2n^2)^2 + 0.01n^2} \frac{4(-1)^{n+1}}{n} \begin{bmatrix} -0.1n \\ 18 - 2n^2 \end{bmatrix}. \end{aligned}$$

The amplitude associated to the frequency $\omega_n = n$ is:

$$\begin{aligned} C_n &= \sqrt{a_n^2 + b_n^2} \\ &= |\frac{1}{(18-2n^2)^2 + 0.01n^2} \frac{4(-1)^{n+1}}{n}| \sqrt{(18-2n^2)^2 + 0.01n^2} \\ &= \frac{4}{n} \frac{1}{\sqrt{(18-2n^2)^2 + 0.01n^2}}. \end{aligned}$$

Here are the first few values:

$$\begin{aligned} C_1 &= 0.2500 \\ C_2 &= 0.2000 \\ C_3 &= 4.4444 \\ C_4 &= 0.0714 \\ C_5 &= 0.0250. \end{aligned}$$

The dominant frequency is $\omega_3 = 3$.