1 Substitute Lecture on March 18, 2011

1.1 Review of Fourier series of functions with period $p = 2\pi$

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $p = 2\pi$. Then we assign to f its Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where the review of formulae for the coefficients a_n and b_n is given below:

Formulae for Fourier coefficients

General case: $a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos mtdt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin mtdt, m = 1, 2, ...$.

Change of interval of integration: $a_m = \frac{1}{\pi} \int_{-\pi+\alpha}^{\pi+\alpha} f(t) \cos mtdt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{\pi} \int_{-\pi+\alpha}^{\pi+\alpha} f(t) \sin mtdt, m = 1, 2, ...$.

In particular, $a_m = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \cos mtdt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \sin mtdt, m = 1, 2, ...$.

If the function is even: $a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos ntdt,\ n = 0, 1, 2, ...$ and
$b_n = 0, n = 1, 2, ...$.

If the function is odd: $a_n = 0,\ n = 0, 1, 2, ...$and
$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin ntdt,\ n = 1, 2, ...$.

It is important to remember that $f(t)$ need not be equal to its Fourier series even when f is a continuous periodic (with period $p = 2\pi$) function.

1.2 General Fourier series

Let $y = f(t)$, be a periodic function with a period $p = 2L$. To obtain Fourier series in this general case consider change of variable:

$$u = \frac{\pi t}{L}.$$

Then if $-L < t < L$, then $-\pi < u < \pi$. We have

$$t = \frac{uL}{\pi}.$$

Set,

$$g(u) = f\left(\frac{uL}{\pi}\right), \text{ where } g \text{ is periodic with } p = 2\pi.$$

We know how to write Fourier series for $g(u)$:

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nu + b_n \sin nu),$$

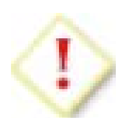
where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u) \cos nudu, \ n = 0, 1, 2, \ldots \text{ and}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u) \sin nudu, \ n = 1, 2, \ldots .$$

In the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(u) \cos nudu,$$

make change of variable:



$$u = \frac{\pi t}{L} \Rightarrow du = \frac{\pi}{L} dtand - L \le t \le L.$$

$$\text{Also recall that } g\left(\frac{\pi t}{L}\right) = f(t).$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u) \cos nudu = \frac{\pi}{L}\frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right) \cos\frac{\pi n t}{L}\frac{\pi}{L} dt$$
$$= \frac{1}{L}\int_{-L}^{L} f(t) \cos\frac{\pi n t}{L} dt, n = 0, 1, 2, \ldots.$$

Likewise,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{\pi n t}{L}dt, n = 1, 2, ...$$

So, we get

$$f(t) = g\left(\frac{\pi t}{L}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi n t}{L} + b_n\sin\frac{\pi n t}{L}\right),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt,\ \ a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{\pi n t}{L}dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{\pi n t}{L}dt,\ \ n = 1, 2, ...$$

Similar remarks as for $p = 2\pi$ are applicable here:

Formulas for Fourier coefficients

General case: $a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{\pi m t}{L}dt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{\pi}\int_{-L}^{L} f(t)\sin\frac{\pi m t}{L}dt, m = 1, 2, ...$

Change the interval of integration: $a_m = \frac{1}{L}\int_{-L+\alpha}^{L+\alpha} f(t)\cos\frac{\pi m t}{L}dt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{L}\int_{-L+\alpha}^{L+\alpha} f(t)\sin\frac{\pi m t}{L}dt, m = 1, 2, ...$

In particular, $a_m = \frac{1}{L}\int_{0}^{2L} f(t)\cos\frac{\pi m t}{L}dt, m = 0, 1, 2, ...,$
$b_m = \frac{1}{L}\int_{0}^{2L} f(t)\sin\frac{\pi m t}{L}dt, m = 1, 2, ...$

If the function is even: $a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\frac{\pi n t}{L}dt,\ n = 0, 1, 2, ...$ and
$b_n = 0, n = 1, 2, ...$

If the function is odd: $a_n = 0,\ n = 0, 1, 2, ...$and
$b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\frac{\pi n t}{L}dt,\ n = 1, 2, ...$

1.3 Convergence of Fourier series

We begin with the following definition:

A function $y = f(t)$ is called piecewise continuous on the segment $[a, b]$ if there are

$$a = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = b$$

such that

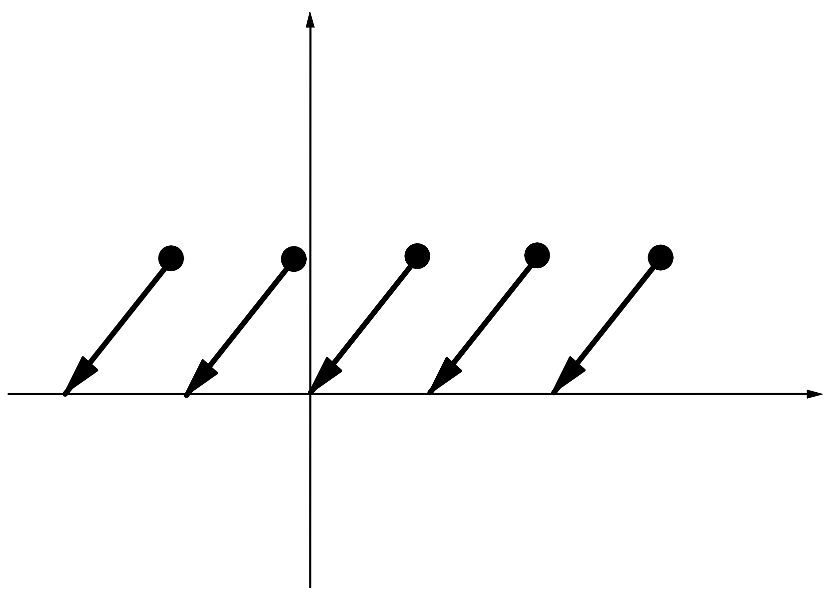
- $f(t)$ is continuous for $t_{j-1} < t < t_j$, where $j = 1, 2, ..., n$.
- The limits

$$\lim_{t \to t_j +} f(t), \lim_{t \to t_j -} f(t), j = 1, 2, ...n-1,$$

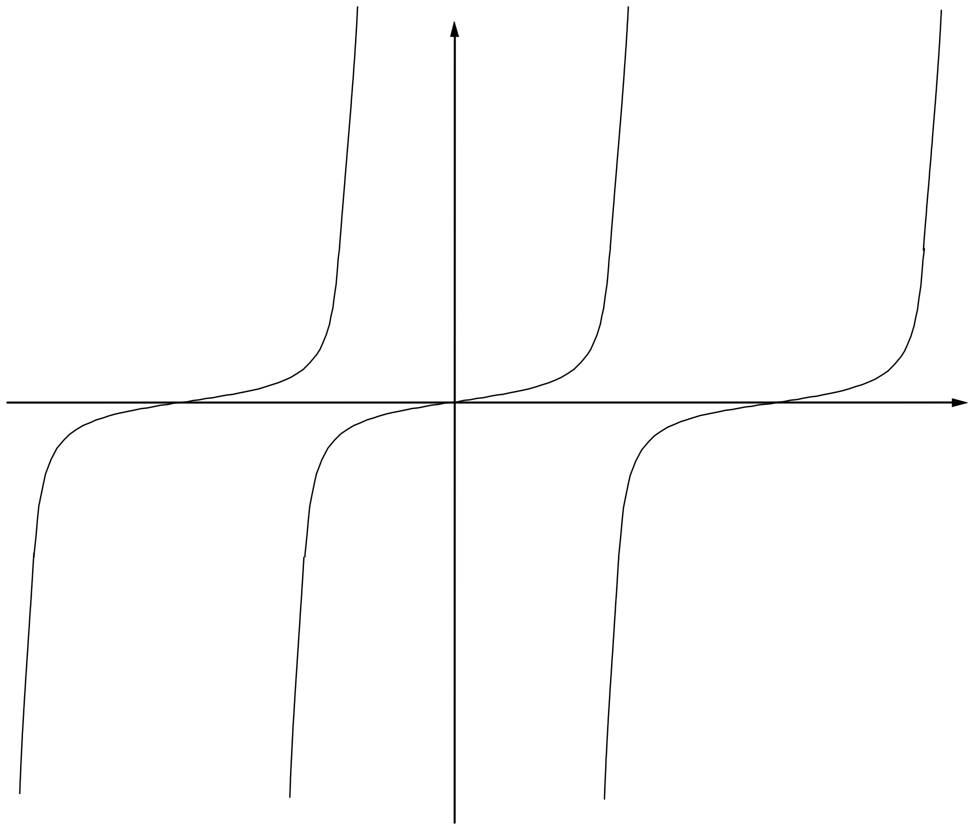
$$\lim_{t \to a+} f(t), \lim_{t \to b-} f(t)$$

  exist and are finite.

For example, the function



is piecewise continuous, whereas the function



is not. Of course, a continuos function is a special case of a piecewise continuous function. If $f(t)$ is defined for every t, then it is piecewise continuous if it is piecewise continuous on every closed segment $[a, b]$. Finally, a piecewise continuous function $f(t)$ is called piecewise smooth if its derivative is piecewise continuous.

We say that f is piecewise continuous on $\mathbb{R}$ if f is piecewise continuous on every segment $[a, b]$; likewise, f is piecewise smooth on $\mathbb{R}$ if f is piecewise smooth on every segment $[a, b]$.

It is important to remember that a continuous function may be different from its Fourier series.

We give without proof the following:

Theorem 1 (Convergence theorem) *Suppose $f(t)$ is periodic with period $p = 2L$. Also, let $f(t)$ be piecewise smooth. Consider Fourier series of $f(t)$:*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n t}{L} + b_n \sin \frac{\pi n t}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t)\, dt, a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{\pi n t}{L} dt,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{\pi n t}{L} dt n = 1, 2, ...$$

Then
(a)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n t}{L} + b_n \sin \frac{\pi n t}{L} \right),$$

if $f(t)$ is continuous at t;
(b)

$$\frac{f(t+0) + f(t-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n t}{L} + b_n \sin \frac{\pi n t}{L} \right),$$

if t is a point of discontinuity of the function $f(t)$.
Recall that

$$f(t+0) = \lim_{x \to t+0} f(x) \text{ and } f(t-0) = \lim_{x \to t-0} f(x).$$

Notice that, as $f(t)$ is piecewise continuous, the foregoing limits exist and are finite. So, for a piecewise smooth periodic function with $p = 2L$,

$$\frac{f(t+0) + f(t-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n t}{L} + b_n \sin \frac{\pi n t}{L} \right),$$

because at the point of continuity

$$\frac{f(t+0) + f(t-0)}{2} = f(t).$$

For a piecewise smooth periodic function $f(t)$ with $p = 2L$ it is customary to write

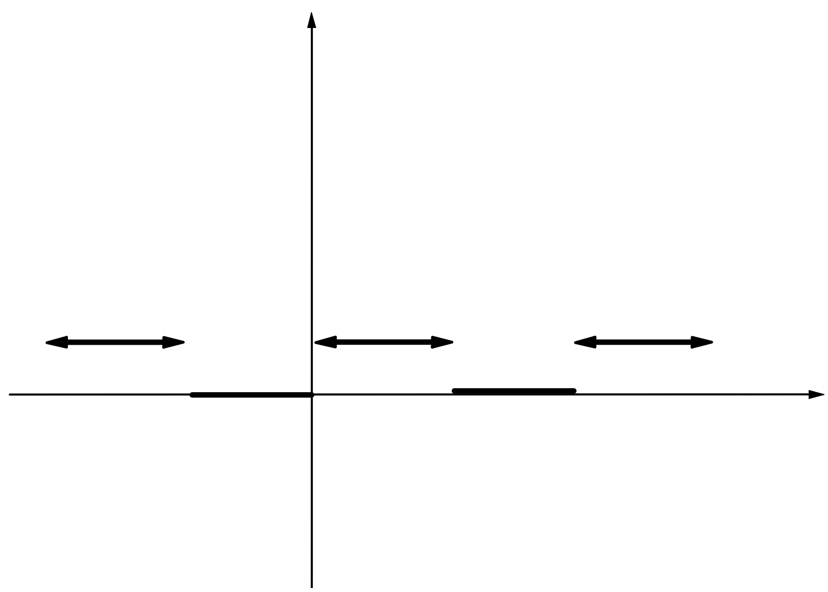
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n t}{L} + b_n \sin \frac{\pi n t}{L} \right),$$

where it goes without saying that instead of values of $f(t)$, at the point of discontinuity, we take the new value:

$$\frac{f(t+0) + f(t-0)}{2}.$$

Example 2

$$f(t) = \begin{cases} 0 & if -5 \leq t \leq 0 \\ 1 & if\, 0 < t < 5 \end{cases}.$$



We have:

$$L = 5, p = 10.$$

$$a_0 = \frac{1}{5} \int_{-5}^{5} f(t)\, dt = \frac{1}{5} \int_{-5}^{0} 0 dt + \frac{1}{5} \int_{0}^{5} dt = 1.$$

$$a_n = \frac{1}{5} \int_{-5}^{5} f(t) \cos \frac{\pi n t}{5} dt = \frac{1}{5} \int_{0}^{5} \cos \frac{\pi n t}{5} dt = \frac{\sin \pi n}{\pi n} = 0, n = 1, 2, ... \,.$$

$$
\begin{aligned}
b_n &= \frac{1}{5}\int_{-5}^{5} f(t)\sin\frac{\pi n t}{5}dt = \frac{1}{5}\int_{0}^{5}\sin\frac{\pi n t}{5}dt = \frac{1}{5}5\frac{1-\cos\pi n}{\pi n} \\
&= \frac{1}{\pi n}\left[1-(-1)^n\right] = \left\{\begin{array}{l} \frac{2}{\pi n} \textit{ if } n \textit{ is odd} \\ 0 \textit{ if } n \textit{ is even} \end{array}\right. .
\end{aligned}
$$

So,

$$
f(t) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{2n-1}{5}\pi t}{2n-1}.
$$

By the convergence theorem,

$$
f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{2n-1}{5}\pi t}{2n-1},
$$

if $t \neq 5k, k = 0, \pm 1, \pm 2, \ldots$ *. If* $t = 5k$*, then*

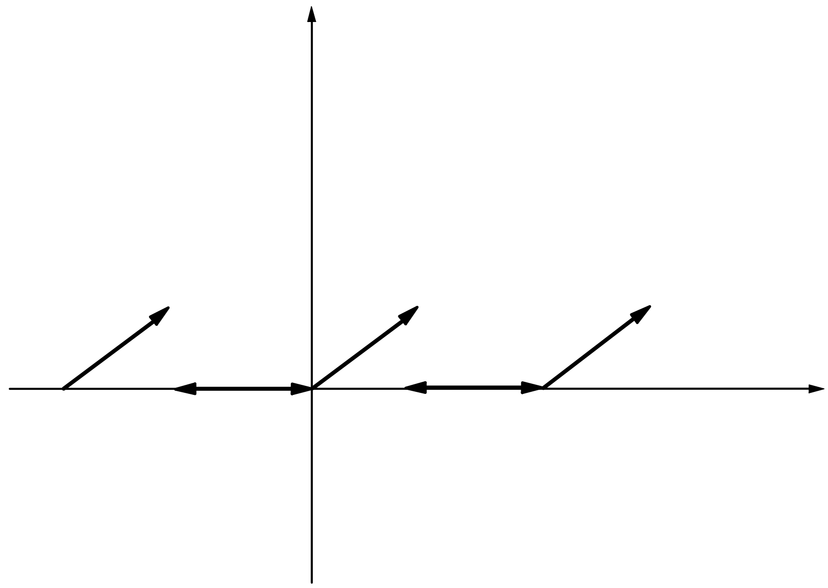
$$
\frac{f(5k+0)+f(5k-0)}{2} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{2n-1}{5}\pi(5k)}{2n-1}.
$$

We can also see this directly:

$$
\begin{aligned}
\sin\frac{2n-1}{5}\pi(5k) &= \sin(2n-1)k\pi = 0 \Rightarrow \\
\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{2n-1}{5}\pi(5k)}{2n-1} &= \frac{1}{2}; \\
\frac{f(5k+0)+f(5k-0)}{2} &= \frac{1+0}{2} = \frac{1}{2}.
\end{aligned}
$$

Example 3

$$
f(t) = \left\{\begin{array}{c} 0 \textit{ if } -1 < t < 0 \\ t \textit{ if } 0 \leq t < 1 \end{array}\right. .
$$



We have $L = 1$ *and* $p = 2$.

$$a_0 = \int_{-1}^{1} f(t)\,dt = \int_0^1 t dt = \frac{1}{2}.$$

$$\begin{aligned} a_n &= \int_{-1}^{1} f(t)\cos \pi n t dt = \int_0^1 t\cos \pi n t dt \\ &= \frac{t\sin \pi n t}{\pi n}|_0^1 - \int_0^1 \frac{\sin \pi n t}{\pi n} dt = \sin \pi n + \frac{\cos \pi n t}{\pi^2 n^2}|_0^1 \\ &= -\frac{1-\cos \pi n}{\pi^2 n^2} = -\frac{1-(-1)^n}{\pi^2 n^2} = \begin{cases} 0 \ \textit{if } n \textit{ is even} \\ -\frac{2}{\pi^2 n^2} \ \textit{if } n \textit{ is odd} \end{cases}. \end{aligned}$$

$$\begin{aligned} b_n &= \int_{-1}^{1} f(t)\sin \pi n t dt = \int_0^1 t\sin \pi n t dt \\ &= -t\frac{\cos \pi n t}{\pi n}|_0^1 + \int_0^1 \frac{\cos \pi n t}{\pi n} dt = -\frac{\cos \pi n}{\pi n} + \frac{\sin \pi n}{\pi^2 n^2} = \frac{(-1)^{n+1}}{\pi n}. \end{aligned}$$

So,

$$f(t) \sim \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos \pi (2n-1) t}{(2n-1)^2} + \frac{1}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin \pi n t}{n}.$$

By the convergence theorem,

$$\begin{aligned}
\frac{f(0+)+f(0-)}{2} &= \frac{0+0}{2} = 0 = \\
&= \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\pi(2n-1)0}{(2n-1)^2} + \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\pi n0}{n} \\
&= \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \Rightarrow \\
\frac{1}{4} &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This example shows that one can use Fourier series to find sums of infinite series with constant coefficients.