Math 285 - Intro Differential Equations
Spring 2011, sections G1 and X1
Review Sheet for the Final

The final exam covers the entire semester, up to the semi-infinite strip in §9.7. The emphasis is on solving equations and on applications, not on the theory.

Topics covered

For a list of topics from midterms 1 and 2, please look at their respective review sheets. Here is a complete list of topics **after midterm 2**.

- §9.5 Heated rod & heat equation $u_t = ku_{xx}$. Solution by separation of variables. Superposition of basic solutions using Fourier coefficients.
- §9.5 Approximate answer as time increases: first term of the series becomes dominant.
- §9.5 Steady-state temperature in the rod: a straight line between the temperatures at the endpoints. Reducing from non-homogeneous to homogeneous boundary conditions – Why is it ok to use the Celcius scale instead of Kelvin?
- §9.5 Insulated endpoints $u_x(0, t) = u_x(L, t) = 0$.
- §9.6 Vibrating string & wave equation $y_{tt} = a^2 y_{xx}$. Solution by separation of variables. Superposition of basic solutions using Fourier coefficients.
- §9.6 Fundamental frequency $\nu_1 = \frac{a}{2L}$.
- §9.6 D'Alembert solution as two waves moving in opposite directions:

$$y(x, t) = \frac{1}{2}\left[F(x + at) + F(a - xt)\right].$$

- §9.7 Steady-state temperature & Laplace equation $u_{xx} + u_{yy} = 0$. Case of a rectangular thin plate. Solution by separation of variables. Superposition of basic solutions using Fourier coefficients.
- §9.7 Reducing from non-homogeneous to homogeneous boundary conditions – Why did we focus on the case where 3 of the 4 sides have temperature 0?
- §9.7 Insulated sides, e.g. left side $u_x(0, y) = 0$ or bottom side $u_y(x, 0) = 0$.
- §9.7 Case of a semi-infinite rectangular strip.

Format

- Two problems on the material from midterm 1.
- Two problems on the material from midterm 2.
- Two long problems on the material after midterm 2, possibly worth as much as the previous 4 problems.

Level of detail required

Since eigenvalue problems and partial differential equations (PDE) have not been tested on the midterms, you will be asked to solve them in detail. How much detail? See the solutions to the practice final for examples of answers that would be deemed complete.

For PDE's: Start from separation of variables. You will come across eigenvalue problems. Solve those explicitly as well.

Example: **Incomplete** work that would **not be accepted**.

$$\begin{aligned} u_t &= k u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= T_0 \end{aligned}$$

The basic solutions are $u_n(x,t) = \exp(-k(\frac{n\pi}{L})^2 t)\sin\frac{n\pi x}{L}$. The initial temperature can be expressed as:

$$T_0 = T_0 \sum_{n\text{ odd}} \frac{4}{n\pi} \sin\frac{n\pi x}{L}.$$

By superposition, we obtain the solution:

$$u(x,t) = T_0 \sum_{n\text{ odd}} \frac{4}{n\pi} \exp(-k(\frac{n\pi}{L})^2 t)\sin\frac{n\pi x}{L}.$$

Why was that answer incomplete?

- Solve the PDE using separation of variables. Who cares if you can remember the basic solutions by heart.
- Solve the eigenvalue problem explicitly.
- Compute the Fourier sine series of the initial temperature explicitly.

For eigenvalue problems: Let's say the equation is $X'' + \lambda X = 0$. **Do treat** the case $\lambda = 0$ separately. Start from the general solution of the equation. Find all eigenvalues and associated eigenfunctions.

You may use the fact that the equation $X'' + \lambda X = 0$ with endpoint conditions of the form $X(a) = X(b) = 0$ or $X'(a) = X'(b) = 0$ or $X(a) = X'(b) = 0$ (for any $a, b \in \mathbb{R}$) does not have negative eigenvalues. It's not terribly hard to show, but it will save you time.

Example: **Incomplete** work that would **not be accepted**.

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(-\pi) = X(\pi) &= 0 \end{aligned}$$

The eigenvalues are $\lambda_n = (\frac{n}{2})^2$ for all positive integers n, with associated eigenfunctions:

$$X_n(x) = \begin{cases} \cos\frac{nx}{2},\ n\text{ odd} \\ \sin\frac{nx}{2},\ n\text{ even.} \end{cases}$$

Why?? How do we know that's true?