**Math 285 - Intro Differential Equations**
**Spring 2011, sections G1 and X1**
**Practice Final, Monday May 2**

**Name:** ________________________________________

**Section (circle):** X1 at 12pm G1 at 3pm

**This is a practice exam, similar in style and difficulty to the real final, but somewhat longer.**

**In the real exam, no calculators, electronic devices, books, or notes may be used.**

**Show your work. No credit for answers without justification.**

**All Fourier series must be computed. Eigenvalue problems must be solved explicitly. Partial differential equations must be solved "from scratch", without quoting formulas.**

**You may omit units for ease of reading, except in final answers which require units.**

**Good luck!**

1. __________/10
2. __________/10
3. __________/10
4. __________/10
5. __________/20
6. __________/20
7. __________/20

Total: __________/100

## A few equations from physics

**What you should know:** Newton's law and Hooke's law.

**1-dimensional heat equation:** $u_t = ku_{xx}$, $k =$ thermal diffusivity (in $\mathrm{cm}^2/\mathrm{s}$)

**2-dimensional heat equation:** $u_t = k\nabla^2 u = k(u_{xx} + u_{yy})$

**Steady-state temperature:** $\nabla^2 u = u_{xx} + u_{yy} = 0$ (Laplace equation)

**1-dimensional wave equation:** $y_{tt} = a^2 y_{xx}$, $a^2 = \frac{T}{\rho}$, $T =$ tension, $\rho =$ density of string.

**Problem 1. (10 pts)** A parachutist weighing $M = 75$ kg jumps out of an airplane $5,000$ meters high and opens his parachute right away. Assume air resistance is proportional to the **square** of the velocity, with coefficient $c = 150\ \mathrm{N/(m/s)^2} = 150\ \mathrm{kg/m}$, so that the parachutist's drag coefficient is $\rho = \frac{c}{M} = 2\mathrm{m}^{-1}$.

Find the **terminal velocity** $v_\tau$ and **how long** it takes for the parachutist to reach a velocity which is 90% of $v_\tau$.

Hint from calculus: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ (for $x \in \mathbb{R}$) and $\frac{d}{dx} \operatorname{arctanh} x = \frac{1}{1-x^2}$ (for $-1 < x < 1$).

**Problem 2. (10 pts)** A 150-liter tank is empty. At time $t = 0$, brine starts pouring in at a flow rate of 4 l/min, and a spigot is turned on so that the (well-mixed) brine in the tank flows **out** at a rate of 1 l/min.

The salt concentration of the incoming brine increases linearly with time: $0.7t$ (g/l)/min. **How much salt is there in the tank once it is full?**

**Problem 3. (10 pts)** Let $f$ denote the $2\pi$-periodic "square wave" function:

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi. \end{cases}$$

Find the **general** solution $x(t)$ of the equation:

$$x'' + 9x = 17f(t) + 5t$$

on the real line.

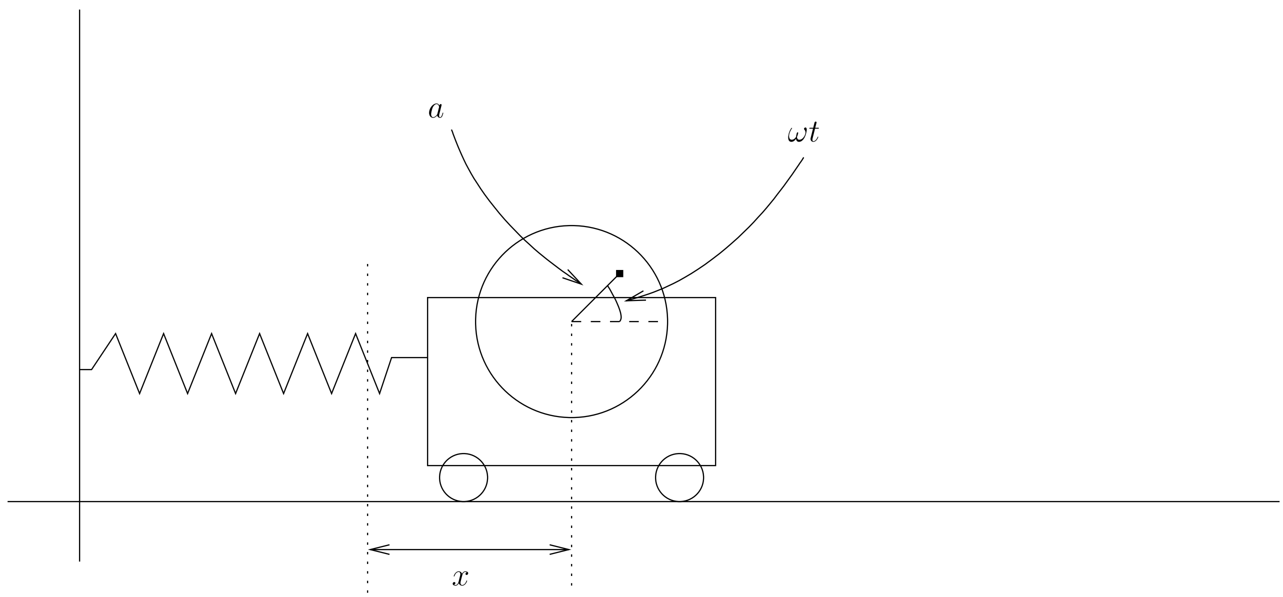


Figure 1: Cart with flywheel attached to a spring.

**Problem 4.** A cart with flywheel is attached to a spring with spring constant 5 N/m (figure 1). The flywheel has a mass $m_0 = 1$ kg and the cart has a mass $m - m_0 = 9$ kg, for a combined mass of $m = 10$ kg.

The center of mass of the flywheel is **not** at the midpoint of the wheel, but rather **off center** by a distance $a = 0.1$ m. However, the midpoint of the wheel is aligned with the center of mass of the cart. The flywheel spins at constant angular speed $\omega = 1$ rad/s.

Assume there is **friction** on the ground proportional to the velocity of the cart, with friction coefficient 2 N/(m/s).

**a. (9 pts)** Let $x(t)$ be the horizontal position of the (center of mass of) the cart, relative to where it would be if the spring were at its equilibrium position. **Find** $x(t)$. (You may use coefficients that depend on the initial conditions.)

(More space)

**b. (1 pt)** Find the steady periodic motion $x_{sp}(t)$ of the cart.

**Problem 5.** A rod of length 30 cm and thermal diffusivity $k$ (in cm$^2$/s) is heated to $100°C$. Then at time $t = 0$, the left end of the rod is embedded in ice at $0°C$ while the right end is kept at $100°C$. (Assume the lateral surface around the rod is insulated.)

**a. (5 pts)** Find the steady-state temperature $u_{ss}(x)$ in the rod.

**b. (15 pts)** Find the temperature $u(x, t)$ in the rod. Hint: First find the transient temperature $u_{tr}(x, t) := u(x, t) - u_{ss}(x)$.

(More space)

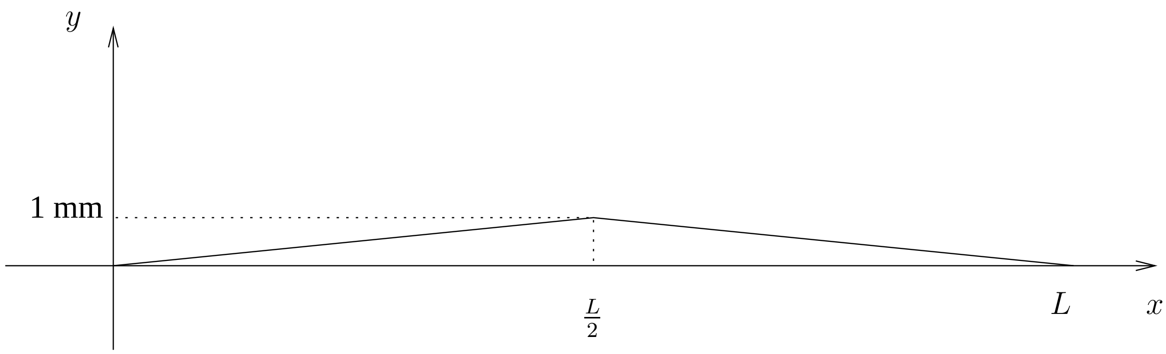


Figure 2: Plucked string.

**Problem 6. (20 pts)** A guitar has strings of length $L = 80$ cm. The second lowest string, which plays a low A (110 Hz), is plucked at its midpoint by pulling it 1 mm and letting it go (figure 2).

Describe the position $y(40, t)$ of the midpoint of the string as a function of time. For example, we know $y(40, 0) = 1$ mm.

(More space)

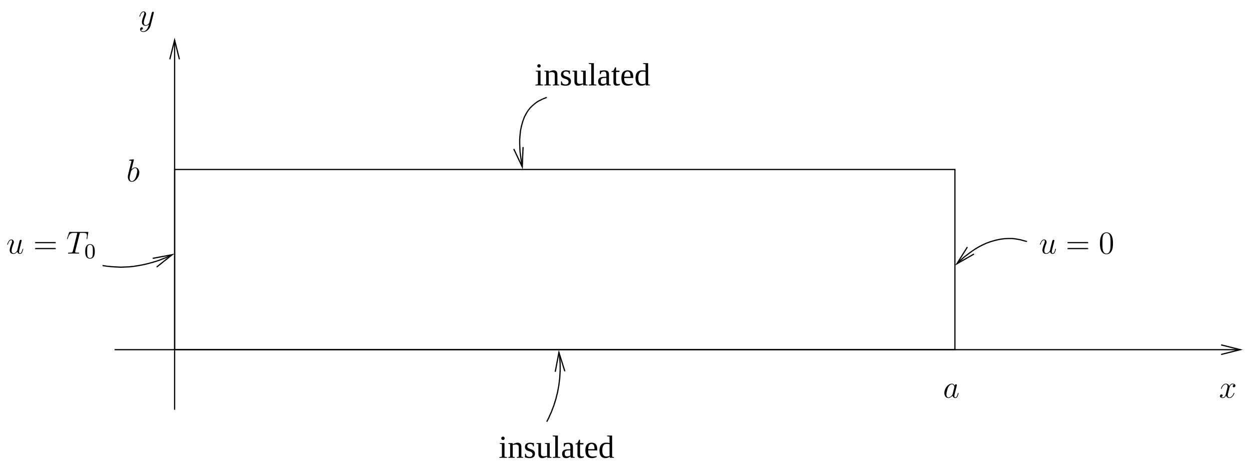


Figure 3: Rectangular thin plate.

**Problem 7.** A thin plate made of aluminum (thermal diffusivity 0.85 $\text{cm}^2/\text{s}$) has the shape of a rectangle of width $a$ and height $b = 10$ cm (figure 3). The plate is sandwiched between insulated sheets, and moreover, the top and bottom edges are insulated. The left edge is kept at constant temperature $T_0 = 43°C$ whereas the right edge is kept at $0°C$.

**a. (10 pts)** Find the **steady-state** temperature $u(x, y)$ in the plate.

(More space)

**b. (6 pts)** Now assume the plate has the shape of a semi-infinite strip, which goes off to infinity to the right. Assuming the temperature remains bounded as $x \to +\infty$, find the **steady-state** temperature $u(x, y)$ in the plate.

**(BACK SIDE)**

**c. (4 pts)** In your answer to part (a), let the width $a$ go to $+\infty$ and check that the result is your answer to part (b).