

Thursday, January 26 ** Worksheet 4 - Projections, distances, and planes.

1. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$.

- (a) Calculate $\text{proj}_{\mathbf{b}}\mathbf{a}$ and draw a picture of it together with \mathbf{a} and \mathbf{b} .
- (b) The orthogonal complement of the vector \mathbf{a} with respect to \mathbf{b} is defined by

$$\text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$

Calculate $\text{orth}_{\mathbf{b}}\mathbf{a}$ and draw two copies of it in your picture from part (a), one based at $\mathbf{0}$ and the other at $\text{proj}_{\mathbf{b}}\mathbf{a}$.

- (c) Check that $\text{orth}_{\mathbf{b}}\mathbf{a}$ calculated in (b) is orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$ calculated in (a).
 - (d) Find the distance of the point $(1, 1)$ from the line $(x, y) = t(2, -1)$. Hint: relate this to your picture.
2. Let \mathbf{a} and \mathbf{b} be vectors in \mathbb{R}^n . Use the definitions of $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ to show that $\text{orth}_{\mathbf{b}}\mathbf{a}$ is always orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$.
3. Find the distance between the point $P(3, 4, -1)$ and the line $\mathbf{l}(t) = (2, 3, -2) + t(1, -1, 1)$. Hint: Consider a vector starting at some point on the line and ending at P , and connect this to what you learned in Problem 1.
4. Consider the equation of the plane $x + 2y + 3z = 12$.

- (a) Find a normal vector to the plane. (Just look at the equation!)
 - (b) Find where the x , y , and z -axes intersect the plane. Using this information, sketch the portion of the plane in the first octant where $x \geq 0, y \geq 0, z \geq 0$.
 - (c) Using the points in part (b), find two non-parallel vectors that are parallel to the plane.
 - (d) Using part (c) and the cross product, find another normal vector to the plane. Show that this vector is parallel to the vector from part (a).
 - (e) Using the new normal vector and one of the points from (b), find an alternative equation for the plane. Compare this new equation to $x + 2y + 3z = 12$. How are these two equations related? Is it clear that they describe the same set of points (x, y, z) in \mathbb{R}^3 ?
5. *The Triangle Inequality.* Let \mathbf{a} and \mathbf{b} be any vectors in \mathbb{R}^n . The triangle inequality states that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.
- (a) Give a geometric interpretation of the triangle inequality. (E.g. draw a picture in \mathbb{R}^2 or \mathbb{R}^3 that represents this inequality.)
 - (b) Use what we know about the dot product to explain why $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$. This is called the Cauchy-Schwarz inequality.
 - (c) Use part (b) to justify the triangle inequality. Hint: Start with the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and then use properties of the dot product and the Cauchy-Schwarz inequality to manipulate the right-hand side into looking like $|\mathbf{a}|^2 + |\mathbf{b}|^2$.