Math 241 - Calculus III
Spring 2012, section CL1
§ 16.9. Gauss's law

In these notes, we discuss Gauss's law and why it is interesting not only for physics, but also from a mathematical viewpoint.

1 Statement

The statement of Gauss's law is as follows. The (net) charge enclosed by a closed surface S is

$$Q = \epsilon_0 \iint_S \vec{E} \cdot \vec{n} \, dS$$

where $\vec{E}$ is the electric field and ϵ_0 is a constant, called the permittivity of free space.

More details can be found in the textbook, § 16.7 after Example 5 and § 16.9 after Example 2.

2 Sketch of proof

Gauss's law follows from Coulomb's law and the divergence theorem.

By Coulomb's law, an electric charge Q at the origin produces the electric field

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3}$$

where $\vec{r} = (x, y, z)$ is the position vector. Such a vector field is sometimes called an inverse square field, because its magnitude

$$|\vec{E}(\vec{r})| = \frac{Q}{4\pi\epsilon_0} \frac{|\vec{r}|}{|\vec{r}|^3} = \frac{Q}{4\pi\epsilon_0} \frac{1}{|\vec{r}|^2}$$

is proportional to the inverse of the square of the distance to the origin (or some other base-point). In symbols: $|\vec{E}(\vec{r})| \propto \frac{1}{|\vec{r}|^2}$.

Step 1: Sphere around the origin

Let S be the sphere of radius R centered at the origin, defined by the equation $x^2+y^2+z^2 = R^2$. Orient S outward, so that the normal vector $\vec{n}$ points away from the origin. The flux of $\vec{E}$ across

S is

$$\begin{aligned}
\iint_S \vec{E} \cdot \vec{n}\, dS &= \iint_S |\vec{E}| dS \;\text{ because } \vec{E} \text{ is parallel to } \vec{n} \\
&= \iint_S \frac{Q}{4\pi\epsilon_0} \frac{1}{|\vec{r}|^2} dS \\
&= \frac{Q}{4\pi\epsilon_0} \iint_S \frac{1}{R^2} dS \\
&= \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \iint_S dS \\
&= \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2}\ \text{Area}(S) \\
&= \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} (4\pi R^2) \\
&= \frac{Q}{\epsilon_0}.
\end{aligned}$$

Step 2: Weird surface around the origin

Now let S' be some arbitrary closed surface enclosing the origin. Orient S' outward. What is the flux of $\vec{E}$ across S'? We can find the answer using the divergence theorem.

Writing $\rho = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ consider the vector field $\vec{F} = \frac{1}{\rho^3}\vec{r} = \frac{1}{\rho^3}(x, y, z)$, which is just $\vec{E}$ scaled by a constant. Noting $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, the divergence is

$$\begin{aligned}
\operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(x\rho^{-3}) + \frac{\partial}{\partial y}(y\rho^{-3}) + \frac{\partial}{\partial z}(z\rho^{-3}) \qquad (1) \\
&= (1)\rho^{-3} + x(-3\rho^{-4})(\frac{x}{\rho}) + (1)\rho^{-3} + y(-3\rho^{-4})(\frac{y}{\rho}) + (1)\rho^{-3} + z(-3\rho^{-4})(\frac{z}{\rho}) \\
&= 3\rho^{-3} - 3\rho^{-5}(x^2 + y^2 + z^2) \\
&= 3\rho^{-3} - 3\rho^{-5}(\rho^2) \\
&= 3\rho^{-3} - 3\rho^{-3} \\
&= 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\operatorname{div} \vec{E} &= \operatorname{div}\left(\frac{Q}{4\pi\epsilon_0}\vec{F}\right) \\
&= \frac{Q}{4\pi\epsilon_0} \operatorname{div} \vec{F} \\
&= 0
\end{aligned}$$

or in words, $\vec{E}$ is incompressible.

By the divergence theorem, the flux across S' is

$$\iint_{S'} \vec{E} \cdot \vec{n}\, dS = \iint_{S} \vec{E} \cdot \vec{n}\, dS$$
$$= \frac{Q}{\epsilon_0}$$

where S is a sphere around the origin (of any radius).

Let us describe the argument in more detail. Pick a giant sphere S which encompasses all of S', and orient S outward. Let D be the solid region between S and S', and orient the boundary of D so that the normal vector points out of D, yielding $\partial D = S - S'$. Applying the divergence theorem to the region D, we obtain

$$\iint_{\partial D} \vec{E} \cdot \vec{n}\, dS = \iiint_{D} \operatorname{div} \vec{E} dV$$
$$= \iiint_{D} 0\, dV$$
$$= 0$$

which can be interpreted as

$$0 = \iint_{\partial D} \vec{E} \cdot \vec{n}\, dS$$
$$= \iint_{S-S'} \vec{E} \cdot \vec{n}\, dS$$
$$= \iint_{S} \vec{E} \cdot \vec{n}\, dS - \iint_{S'} \vec{E} \cdot \vec{n}\, dS.$$

In other words, the flux across S is the same as the flux across S', as claimed above.

Note that the divergence theorem does **not** apply to the region enclosed by S, i.e. the punctured solid ball defined by $0 < x^2 + y^2 + z^2 \leq R^2$, because that region is not closed. The singularity at the origin prevents us from using the divergence theorem.

Step 3: Weird surface not around the origin

Now what if S'' is a closed surface that does **not** enclose the origin? Then S'' is the boundary of a solid region D'' which does not contain the origin, and the divergence theorem applies:

$$\iint_{S''} \vec{E} \cdot \vec{n}\, dS = \iint_{\partial D''} \vec{E} \cdot \vec{n}\, dS$$
$$= \iiint_{D''} \operatorname{div} \vec{E} dV$$
$$= \iiint_{D''} 0\, dV$$
$$= 0.$$

In short, we have shown that if S is a closed surface (with outward orientation), then the flux of the electric field $\vec{E}$ across S is

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \begin{cases} \frac{Q}{\epsilon_0} & \text{if } S \text{ encloses the origin} \\ 0 & \text{if } S \text{ does not enclose the origin.} \end{cases}$$

We can rewrite this as

$$\epsilon_0 \iint_S \vec{E} \cdot \vec{n}\, dS = \begin{cases} Q & \text{if } S \text{ encloses the origin} \\ 0 & \text{if } S \text{ does not enclose the origin} \end{cases}$$

which is equal to the (net) charge enclosed by S. This proves Gauss's law in the case of a single (pointlike) charge.

Step 4: Many electric charges

For a finite system of charges Q_i at positions $\vec{r}_i$, consider the electric field $\vec{E}_i(\vec{r}) = \frac{Q_i}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$ produced by each charge. The total electric field $\vec{E}$ is their superposition:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N.$$

For any closed surface S (oriented outward), ϵ_0 times the flux of the electric field $\vec{E}$ across S is

$$\begin{aligned} \epsilon_0 \iint_S \vec{E} \cdot \vec{n}\, dS &= \epsilon_0 \iint_S \left(\sum_{i=1}^{N} \vec{E}_i \right) \cdot \vec{n}\, dS \\ &= \sum_{i=1}^{N} \epsilon_0 \iint_S \vec{E}_i \cdot \vec{n}\, dS \\ &= \sum_{\substack{i \text{ such that } S \text{ encloses} \\ \text{the position } \vec{r}_i}} Q_i \\ &= \text{net charge enclosed by } S. \end{aligned}$$

This proves Gauss's law in the case of finitely many (pointlike) charges.

A similar argument proves Gauss's law in the case of a continuous distribution of electric charges, described by a charge density function.

3 Again, which vector fields are curls?

In section § 16.8, we asked the question: How do we know if a vector field $\vec{F}$ is the curl of some vector field $\vec{G}$? We found a necessary condition: a curl is always incompressible, i.e.

$$\operatorname{div}(\operatorname{curl} \vec{G}) \equiv 0.$$

Then we wondered if that condition is sufficient: Given $\operatorname{div} \vec{F} = 0$, can we conclude that $\vec{F}$ is the curl of some vector field? We provided the answer – **NO!** – without justification. Now we can justify that negative answer.

Consider the inverse square vector field $\vec{F} = \frac{1}{\rho^3}\vec{r} = \frac{1}{\rho^3}(x, y, z)$. It is incompressible, i.e.

$$\operatorname{div} \vec{F} = 0$$

as computed in (1). However, $\vec{F}$ is **not** the curl of a vector field. Indeed, we have found a closed surface S (say, a sphere centered at the origin) such that the flux of $\vec{F}$ across S is non-zero:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = 4\pi \neq 0.$$

Therefore $\vec{F}$ is not a curl.

Here we used Stokes' theorem, which implies that the flux of $\operatorname{curl} \vec{G}$ across any closed surface must be zero:

$$\begin{aligned} \iint_S \operatorname{curl} \vec{G} \cdot \vec{n}\, dS &= \int_{\partial S} \vec{G} \cdot d\vec{r} \\ &= \int_{\emptyset} \vec{G} \cdot d\vec{r} \\ &= 0. \end{aligned} \tag{2}$$

Remark 3.1. With a bit of topology, one can show that property (2) characterizes curls: A vector field is a curl **if and only if** its flux across any closed surface is zero.

Recall that there is a partial converse. The condition of being incompressible is sometimes sufficient for being a curl.

Proposition 3.2. *Let $\vec{F}$ be a continuously differentiable vector field* ***on all of*** $\mathbb{R}^3$*. If $\vec{F}$ satisfies $\operatorname{div} \vec{F} \equiv 0$, then $\vec{F}$ is the curl of some vector field. In words: a vector field on all of $\mathbb{R}^3$ is a curl if and only if it is incompressible.*

Proof. The key point is that in $\mathbb{R}^3$, a closed surface S always bounds a solid region D.

Let $\vec{F}$ be a vector field satisfying $\operatorname{div} \vec{F} \equiv 0$ and let S be any closed surface. Let D be the solid region bounded by S. Then the flux of $\vec{F}$ across S (oriented outward) is

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n}\, dS &= \iint_{\partial D} \vec{F} \cdot \vec{n}\, dS \\ &= \iiint_D \operatorname{div} \vec{F}\, dV \\ &= \iiint_D 0\, dV \\ &= 0. \end{aligned}$$

By 3.1, $\vec{F}$ is the curl of some vector field. □