Math 241 - Calculus III
Spring 2012, section CL1
§ 16.8. Stokes' theorem

In these notes, we illustrate Stokes' theorem by a few examples, and highlight the fact that many different surfaces can bound a given curve.

1 Statement of Stokes' theorem

Let S be a surface in $\mathbb{R}^3$ and let ∂S be the boundary (curve) of S, oriented according to the usual convention. That is, "if we move along ∂S and fall to our left, we hit the side of the surface where the normal vectors are sticking out". Let $\vec{F}$ be a vector field that is defined (and smooth) in a neighborhood of S. Then the following equality holds:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS.$$

The theorem can be useful in either direction: sometimes the line integral is easier than the surface integral, sometimes the other way around.

2 Examples

Consider the vector field $\vec{F} = (y, xz, 1)$ whose curl is

$$\begin{aligned}
\operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & 1 \end{vmatrix} \\
&= \vec{i}(0 - x) - \vec{j}(0 - 0) + \vec{k}(z - 1) \\
&= (-x, 0, z - 1).
\end{aligned}$$

Consider the curve C which is the unit circle in the xy-plane, defined by $x^2 + y^2 = 1$, $z = 0$, oriented counterclockwise when viewed from above. Let us compute the line integral of $\vec{F}$ along C.

First, parametrize C by the usual "longitude" angle θ:

$$\begin{aligned}
&\vec{r}(\theta) = (\cos\theta, \sin\theta, 0) \\
&0 \leq \theta \leq 2\pi.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (y, xz, 1) \cdot \vec{r}\,'(t) dt \\
&= \int_0^{2\pi} (\sin\theta, 0, 1) \cdot (-\sin\theta, \cos\theta, 0) dt \\
&= \int_0^{2\pi} -\sin^2\theta dt \\
&= \boxed{-\pi}.
\end{aligned}$$

Stokes' theorem claims that if we "cap off" the curve C by any surface S (with appropriate orientation) then the line integral can be computed as

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS.$$

Now let's have fun! More precisely, let us verify the claim for various choices of surface S.

2.1 Disk

Take S to be the unit disk in the xy-plane, defined by $x^2 + y^2 \leq 1,\ z = 0$. According to the orientation convention, the normal $\vec{n}$ to S should be oriented upward, so that in fact $\vec{n} = (0, 0, 1)$.

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_S (-x, 0, z-1) \cdot (0,0,1) dS \\
&= \iint_S (z-1) dS \\
&= \iint_S (-1) dS \\
&= -\text{Area}(S) \\
&= \boxed{-\pi}.
\end{aligned}$$

2.2 Hemisphere

Take S to be the unit **upper hemisphere**, defined by $x^2+y^2+z^2 = 1,\ z \geq 0$. According to the orientation convention, the normal $\vec{n}$ to S should be oriented upward, pointing away from the origin. That means $\vec{n} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x, y, z)$. Let us parametrize S in spherical coordinates, with colatitude $0 \leq \phi \leq \frac{\pi}{2}$ and longitude $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_S (-x, 0, z-1) \cdot (x, y, z) dS \\
&= \iint_S -x^2 + z^2 - z\, dS \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(-(\sin\phi\cos\theta)^2 + (\cos\phi)^2 - \cos\phi\right) \sin\phi\, d\theta d\phi \\
&= \int_0^{\frac{\pi}{2}} (-\pi \sin^2\phi + 2\pi\cos^2\phi - 2\pi\cos\phi) \sin\phi\, d\phi \\
&= \pi \int_0^{\frac{\pi}{2}} (\cos^2\phi - 1 + 2\cos^2\phi - 2\cos\phi) \sin\phi\, d\phi \\
&= \pi \int_0^{\frac{\pi}{2}} (3\cos^2\phi - 2\cos\phi - 1) \sin\phi\, d\phi \qquad u = \cos\phi, du = -\sin\phi d\phi \\
&= \pi \int_1^0 (3u^2 - 2u - 1)(-du) \\
&= \pi \left[u^3 - u^2 - u\right]_0^1 \\
&= \pi(1 - 1 - 1) \\
&= \boxed{-\pi}.
\end{aligned}$$

We could also take S to be the unit **lower hemisphere**, defined by $x^2 + y^2 + z^2 = 1$, $z \leq 0$. According to the orientation convention, the normal $\vec{n}$ to S should be oriented upward, pointing towards the origin. That means $\vec{n} = \frac{-(x,y,z)}{\sqrt{x^2+y^2+z^2}} = -(x, y, z)$. Again, we parametrize S in spherical coordinates, with colatitude $\frac{\pi}{2} \leq \phi \leq \pi$ and longitude $0 \leq \theta \leq 2\pi$. A very similar calculation yields:

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_S (-x, 0, z-1) \cdot (-(x, y, z))\, dS \\
&= -\iint_S -x^2 + z^2 - z\, dS \\
&= -\pi \int_{\frac{\pi}{2}}^{\pi} (3\cos^2\phi - 2\cos\phi - 1) \sin\phi\, d\phi \\
&= -\pi \int_0^{-1} (3u^2 - 2u - 1)(-du) \\
&= \pi \left[u^3 - u^2 - u\right]_0^{-1} \\
&= \pi\left((-1) - 1 - (-1)\right) \\
&= \boxed{-\pi}.
\end{aligned}$$

2.3 Paraboloid

Take S to be the part of the paraboloid defined by $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$. According to the orientation convention, the normal $\vec{n}$ to S should be oriented upward, pointing away from the z-axis.

Let us parametrize S using x and y as parameters, with domain of parametrization D the unit disk $x^2 + y^2 \leq 1$:

$$\vec{r}(x, y) = (x, y, 1 - x^2 - y^2).$$

The tangent vectors are

$$\begin{aligned} \vec{r}_x &= (1, 0, -2x) \\ \vec{r}_y &= (0, 1, -2y) \end{aligned}$$

so that a normal vector is given by the cross product

$$\vec{r}_x \times \vec{r}_y = (2x, 2y, 1).$$

Check the orientation: This normal vector points up and away from the z-axis. Ok! No need to flip it.

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_D \operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) dxdy \\ &= \iint_D (-x, 0, z-1) \cdot (2x, 2y, 1) dxdy \\ &= \iint_D -2x^2 + z - 1 dxdy \\ &= \iint_D -2x^2 + (1 - x^2 - y^2) - 1 dxdy \\ &= \iint_D -3x^2 - y^2 dxdy \\ &= -\int_0^{2\pi} \int_0^1 \left(3r^2 \cos^2\theta + r^2 \sin^2\theta\right) r dr d\theta \\ &= -\int_0^{2\pi} \int_0^1 \left(3r^3 \cos^2\theta + r^3 \sin^2\theta\right) dr d\theta \\ &= -\int_0^{2\pi} \frac{3}{4} \cos^2\theta + \frac{1}{4} \sin^2\theta d\theta \\ &= -\left(\frac{3}{4}(\pi) + \frac{1}{4}(\pi)\right) \\ &= \boxed{-\pi}. \end{aligned}$$

2.4 Cone

Take S to be the part of the cone defined by $z = 1 - \sqrt{x^2 + y^2}$, $x^2 + y^2 \leq 1$. According to the orientation convention, the normal $\vec{n}$ to S should be oriented upward, pointing away from the

z-axis.

Let us parametrize S in cylindrical coordinates with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$:

$$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, 1-r).$$

The tangent vectors are

$$\begin{aligned}\vec{r}_r &= (\cos\theta, \sin\theta, -1)\\ \vec{r}_\theta &= (-r\sin\theta, r\cos\theta, 0)\end{aligned}$$

so that a normal vector is given by the cross product

$$\begin{aligned}\vec{r}_r \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\ &= \vec{i}(0-(-r\cos\theta)) - \vec{j}(0 - r\sin\theta) + \vec{k}(r\cos^2\theta - (-r\sin^2\theta)) \\ &= (r\cos\theta, r\sin\theta, r).\end{aligned}$$

Check the orientation: This normal vector points up and away from the z-axis. Ok! No need to flip it.

$$\begin{aligned}\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS &= \iint_D \operatorname{curl}\vec{F}\cdot(\vec{r}_r\times\vec{r}_\theta)\,drd\theta \\ &\quad \text{Notice! The vector } \vec{r}_r\times\vec{r}_\theta = (r\cos\theta, r\sin\theta, r) \text{ already encodes the distortion} \\ &\quad \text{of area } dS = |\vec{r}_r\times\vec{r}_\theta|\,drd\theta = \sqrt{2}\,r\,drd\theta\text{. Do not throw in an extra } r. \\ &= \iint_D (-x, 0, z-1)\cdot(r\cos\theta, r\sin\theta, r)drd\theta \\ &= \iint_D \left(-r^2\cos^2\theta + (-r)r\right)drd\theta \\ &= -\int_0^{2\pi}\int_0^1 r^2(\cos^2\theta+1)drd\theta \\ &= -\frac{1}{3}\int_0^{2\pi}(\cos^2\theta+1)d\theta \\ &= -\frac{1}{3}(\pi+2\pi) \\ &= \boxed{-\pi}.\end{aligned}$$

2.5 Tin can

Take S to be the "tin can with a top but without the bottom", of height, say, 5. In other words, S consists of the part of the cyclinder $x^2+y^2=1$, $0 \leq z \leq 5$ (the side of the can), along with the disk $x^2+y^2 \leq 1$, $z=5$ (the top of the can). Call them respectively S_1 and S_2.

According to the orientation convention, the normal $\vec{n}$ to S_1 should point away from the z-axis. That means $\vec{n} = (x, y, 0)$. Let us parametrize S_1 in cylindrical coordinates with $0 \le z \le 5$ and $0 \le \theta \le 2\pi$:

$$\vec{r}(z, \theta) = (\cos\theta, \sin\theta, z).$$

$$\begin{aligned}
\iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_{S_1} (-x, 0, z-1) \cdot (x, y, 0) dS \\
&= \iint_{S_1} -x^2\, dS \\
&= -\int_0^{2\pi} \int_0^5 (\cos^2\theta)(1) dz d\theta \\
&= -5 \int_0^{2\pi} (\cos^2\theta) d\theta \\
&= -5\pi.
\end{aligned}$$

The normal $\vec{n}$ to S_2 should point up. That means $\vec{n} = (0, 0, 1)$.

$$\begin{aligned}
\iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_{S_2} (-x, 0, z-1) \cdot (0, 0, 1) dS \\
&= \iint_{S_2} (z-1) dS \\
&= \iint_{S_2} (4) dS \\
&= 4 \operatorname{Area}(S_2) \\
&= 4\pi.
\end{aligned}$$

Combining the two parts, we obtain

$$\begin{aligned}
\iint_{S} \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n}\, dS + \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n}\, dS \\
&= -5\pi + 4\pi \\
&= \boxed{-\pi}.
\end{aligned}$$

3 Closed surfaces

3.1 Closed curves

Recall the following from chapter 13.

Definition 3.1. A **closed curve** is a curve that ends where it started.

In other words, a closed curve C has no endpoints floating around; it forms a loop. Another way to say this is that its boundary is empty: $\partial C = \emptyset$. In general, the boundary of a curve is its ending point minus its starting point (the signs account for orientation). If the curve C goes from A to B, then we can write

$$\partial C = B - A.$$

A consequence of the fundamental theorem of line integrals is that integrating a **conservative** vector field along a **closed** curve C automatically yields zero:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 0. \tag{1}$$

In fact, we learned that property (1) characterizes conservative vector fields: A vector field is conservative **if and only if** its integral along any loop is zero.

Property (1) is not as mysterious as it seems. The key is that conservative vector fields are very special. Most vector fields are not conservative, i.e. are not the gradient of any function.

3.2 In 2 dimensions

What is the analogous notion for 2-dimensional objects, namely surfaces?

Definition 3.2. A **closed surface** is a surface that has no boundary.

In other words, a closed surface S has no "edge" floating around. Another way to say this is that its boundary is empty: $\partial S = \emptyset$. In general, the boundary of a surface will be a curve, or possibly several curves.

Example 3.3. Let S is the upper hemisphere of radius R, defined by $x^2 + y^2 + z^2 = R^2$, $z \geq 0$. Its boundary ∂S is the circle of radius R in the xy-plane, defined by $x^2 + y^2 = R^2$, $z = 0$.

Example 3.4. Let S is the sphere of radius R, defined by $x^2 + y^2 + z^2 = R^2$. Its boundary ∂S is empty. That is, the sphere is a **closed** surface.

Example 3.5. Let S is the part of the cylinder of radius R around the z-axis, of height H, defined by $x^2 + y^2 = R^2$, $0 \leq z \leq H$. Its boundary ∂S consists of two circles of radius R: C_1 defined by $x^2 + y^2 = R^2$, $z = 0$, and C_2 defined by $x^2 + y^2 = R^2$, $z = H$.

A consequence of Stokes' theorem is that integrating a vector field which is a **curl** along a **closed** surface S automatically yields zero:

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \int_{\partial S} \vec{F} \cdot d\vec{r} \\ &= \int_{\emptyset} \vec{F} \cdot d\vec{r} \\ &= 0. \end{aligned} \tag{2}$$

Remark 3.6. In case the idea of integrating over an empty set feels uncomfortable – though it shouldn't – here is another way of thinking about the statement. If S is a closed surface, cut it into two parts S_1 and S_2 along some curve C. For example, we can cut the sphere S into the upper hemisphere S_1 and lower hemisphere S_2 along the equator C. Applying Stokes's theorem to each part yields

$$\begin{aligned}
\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS &= \iint_{S_1} \operatorname{curl}\vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \operatorname{curl}\vec{F}\cdot\vec{n}\,dS \\
&= \int_C \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r} \\
&= 0
\end{aligned}$$

where the opposite signs come from the orientation convention.

In fact, property (2) characterizes curls: A vector field is the curl of some vector field **if and only if** its integral along any closed surface is zero.

Property (2) is not as mysterious as it seems. The key is that curls are very special. Most vector fields are not the curl of a vector field.

4 Which vector fields are curls?

We have seen that vector fields of the form $\operatorname{curl}\vec{F}$ are (relatively) easy to integrate along surfaces. But how do we know if a given vector field is the curl of some vector field? Here is a necessary condition.

Proposition 4.1. *Let $\vec{F}$ be a nice enough vector field (twice continuously differentiable). Then we have* $\operatorname{div}(\operatorname{curl}\vec{F}) \equiv 0$. *In words: a curl is always incompressible.*

Proof. Write $\vec{F} = (F_1, F_2, F_3)$ and abbreviate the partial differentiation operators as

$$\partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y},\ \partial_3 = \frac{\partial}{\partial z}.$$

Then the curl is

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \vec{i}(\partial_2 F_3 - \partial_3 F_2) - \vec{j}(\partial_1 F_3 - \partial_3 F_1) + \vec{k}(\partial_1 F_2 - \partial_2 F_1) \\
&= (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1).
\end{aligned}$$

Its divergence is

$$\begin{aligned}
\operatorname{div}\left(\operatorname{curl}\vec{F}\right) &= \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) \\
&= \partial_1\partial_2 F_3 - \partial_1\partial_3 F_2 + \partial_2\partial_3 F_1 - \partial_2\partial_1 F_3 + \partial_3\partial_1 F_2 - \partial_3\partial_2 F_1 \\
&= (\partial_1\partial_2 F_3 - \partial_2\partial_1 F_3) + (\partial_2\partial_3 F_1 - \partial_3\partial_2 F_1) + (\partial_3\partial_1 F_2 - \partial_1\partial_3 F_2) \\
&\equiv 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

□

Remark 4.2. If one prefers the notation $\vec{F} = (P, Q, R)$, then the above calculations can be written as

$$\operatorname{curl} \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

and

$$\operatorname{div}\left(\operatorname{curl} \vec{F}\right) = (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) \equiv 0.$$

Example 4.3. We have seen in section 2 that the vector field $(-x, 0, z-1)$ is a curl. Its divergence is indeed zero:

$$\operatorname{div}(-x, 0, z-1) \equiv -1 + 0 + 1 = 0.$$

Example 4.4. Consider the vector field $\vec{F} = (xy^2, y + z, yz^3)$. Is $\vec{F}$ the curl of some vector field?

The divergence is

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(yz^3) \\ &= y^2 + 1 + 3yz^2 \end{aligned}$$

which is not the constant function 0. Therefore $\vec{F}$ is **not** the curl of a vector field.

Question 4.5. Is the condition also sufficient? In other words, does the property $\operatorname{div} \vec{F} \equiv 0$ guarantee that $\vec{F}$ is a curl?

In general, the answer is **NO**! However, there is a partial converse.

Proposition 4.6. *Let $\vec{F}$ be a nice enough vector field* ***on all of*** $\mathbb{R}^3$*. If $\vec{F}$ satisfies $\operatorname{div} \vec{F} \equiv 0$, then $\vec{F}$ is the curl of some vector field. In words: a vector field defined everywhere on $\mathbb{R}^3$ is a curl if and only if it is incompressible.*

Example 4.7. Consider the vector field $\vec{F} = (xy^2, -y^3 + \cos z, 2y^2 z)$. Is $\vec{F}$ the curl of some vector field?

The divergence is

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-y^3 + \cos z) + \frac{\partial}{\partial z}(2y^2 z) \\ &= y^2 - 3y^2 + 2y^2 \\ &\equiv 0. \end{aligned}$$

Moreover, $\vec{F}$ is defined (and smooth) everywhere on $\mathbb{R}^3$. Therefore $\vec{F}$ **is** the curl of some vector field.

Remark 4.8. Describing more precise sufficient conditions for an incompressible vector field to be a curl would require a foray into topology. The answer depends on the shape of the domain of $\vec{F}$.

In the lecture on Monday April 30, we will see that the converse of 4.1 can fail spectacularly. We will study a vector field $\vec{F}$ satisfying $\operatorname{div} \vec{F} \equiv 0$ which is not a curl.